

POINCARÉ DUALITY ISOMORPHISMS IN TENSOR CATEGORIES

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ABSTRACT. If for a vector space V of dimension g over a characteristic zero field we denote by $\wedge^i V$ its alternating powers, and by V^\vee its linear dual, then there are natural Poincaré isomorphisms:

$$\wedge^i V^\vee \cong \wedge^{g-i} V.$$

We describe an analogous result for objects in rigid pseudo-abelian \mathbb{Q} -linear ACU tensor categories.

CONTENTS

1. Introduction	1
2. Linear algebra in tensor categories	4
3. A formal Poincaré duality isomorphism	16
4. Application to Δ -graded algebras in \mathcal{C}	22
5. A Poincaré duality isomorphism for the alternating algebras	24
6. A Poincaré duality isomorphism for the symmetric algebras	35
References	37

1. INTRODUCTION

Let V be a vector space of finite dimension g over a characteristic zero field, let \mathbb{I} be the field of scalars, viewed as a vector space, and consider the alternating algebra $\wedge^\bullet V$. Then the internal multiplication morphism defined by the formula

$$\iota_1(x)(\omega_1 \wedge \dots \wedge \omega_j) := \sum_{k=1}^j (-1)^j \langle x, \omega_k \rangle \omega_1 \wedge \dots \wedge \widehat{\omega}_k \wedge \dots \wedge \omega_j$$

gives a map

$$\iota_1 : V \rightarrow \text{Hom}(\wedge^\bullet V^\vee, \wedge^{\bullet-1} V^\vee)$$

valued in the space of degree -1 anti-derivation. Since $\iota_1(x)^2 = 0$, by the universal property of the alternating algebra the morphism ι_1 extends to a morphism of algebras

$$\iota : \wedge^\bullet V \rightarrow \text{Hom}(\wedge^\bullet V^\vee, \wedge^\bullet V^\vee)^{op},$$

where $(\cdot)^{op}$ means the opposite algebra, such that $\iota(x) : \wedge^j V^\vee \rightarrow \wedge^{j-i} V^\vee$ if $x \in \wedge^i V$ and $j \geq i$ (and it is zero otherwise). In order to match with the notations employed in the paper, it will be convenient to define, for every $j \geq i$:

$$\iota_{i,j}(x) := \frac{(j-i)!}{j!} \iota(x)|_{\wedge^j V^\vee} : \wedge^j V^\vee \rightarrow \wedge^{j-i} V^\vee, \text{ if } x \in \wedge^i V.$$

This gives morphisms $\iota_{i,j} : \wedge^i V \rightarrow \text{Hom}(\wedge^j V^\vee, \wedge^{j-i} V^\vee)$ with the following property. If we identify $\wedge^\bullet V^\vee \simeq (\wedge^\bullet V)^\vee$ by means of

$$\text{ev}_{V,a}^i : \wedge^i V^\vee \otimes \wedge^i V \rightarrow \mathbb{I} \quad (1)$$

obtained by the natural inclusions $\wedge^i(\cdot) \hookrightarrow \otimes^i(\cdot)$ followed by the perfect pairing

$$\text{ev}_V^i(\omega_1 \otimes \dots \otimes \omega_i, x_1 \otimes \dots \otimes x_i) := \prod_{k=1}^i \langle \omega_k, x_k \rangle,$$

then

$$\text{ev}_{V,a}^j(\omega_j, x_i \wedge x_{j-i}) = \text{ev}_{V,a}^{j-i}(\iota_{i,j}(x_i)(\omega_j), x_{j-i}) \text{ for } x_i \in \wedge^i V, x_{j-i} \in \wedge^{j-i} V^\vee \text{ and } \omega_j \in \wedge^j V^\vee, \quad (2)$$

meaning that $\iota_{i,j}(x_i) : \wedge^j V^\vee \rightarrow \wedge^{j-i} V^\vee$ is dual to the multiplication map $x_i \wedge \cdot : \wedge^{j-i} V \rightarrow \wedge^j V$.

These internal multiplication morphisms allow for the definition of the Poincaré morphism

$$D^{i,g} : \wedge^i V \xrightarrow{\iota_{i,g}} \text{Hom}(\wedge^g V^\vee, \wedge^{g-i} V^\vee) \simeq \wedge^{g-i} V^\vee$$

and using reflexivity after dualizing yields

$$D_{i,g} : \wedge^i V^\vee \xrightarrow{\iota_{i,g}} \text{Hom}(\wedge^g V, \wedge^{g-i} V) \simeq \wedge^{g-i} V.$$

As it is well known one has

$$D_{g-i,g} \circ D^{i,g} = (-1)^{i(g-i)} \binom{g}{g-i}^{-1} \text{ and } D^{i,g} \circ D_{g-i,g} = (-1)^{i(g-i)} \binom{g}{i}^{-1} \quad (3)$$

If the category of finite dimensional vector spaces is replaced by a more general neutral tannakian category, the fibre functor allows to extend this result to this category due to (3) and the existence of a faithful exact linear functor valued in the category of vector spaces, once the appropriate definition of the Poincaré morphism is given in such a way that it is preserved by tensor functors. The aim of this paper is to generalize this result to rigid pseudo-abelian and \mathbb{Q} -linear ACU tensor categories, with the aim of applications to Chow motives, and prove the analogue statement for the symmetric algebras $\vee V$.

Suppose indeed that V is a supervector space of odd degree. Then the same formalism applies, replacing the alternating algebra with the symmetric algebra: the reason is that, by definition, the commutativity constraint $\tau_{V,W} : V \otimes W \rightarrow W \otimes V$ in the category of supervector spaces is given by $\tau_{V,W}(x \otimes y) = -(y \otimes x)$ if V and W have odd degree and, hence, the symmetrizer operates as an anti-symmetrizer on the underlying vector spaces.

The viewpoint taken in this paper is to use (2) as the defining property of the internal multiplication morphisms. Suppose that \mathcal{C} is a rigid pseudo-abelian and \mathbb{Q} -linear ACU tensor category with identity object \mathbb{I} and that we are given $V \in \mathcal{C}$ of rank $r \in \text{End}(\mathbb{I})$. If A_i denotes one of the alternating or symmetric algebras, the data of the multiplication morphisms $\varphi_{i,j} : A_i \otimes A_j \rightarrow A_{i+j}$ is equivalent to that of the associated morphisms $f_{i,j} : A_i \rightarrow \text{hom}(A_j, A_{i+j})$. When $j \geq i$, we may consider the composite

$$\iota_{i,j} : A_i \xrightarrow{f_{i,j-i}} \text{hom}(A_{j-i}, A_j) \xrightarrow{d} \text{hom}(A_j^\vee, A_{j-i}^\vee)$$

where $d : \text{hom}(X, Y) \rightarrow \text{hom}(Y^\vee, X^\vee)$ is the internal duality morphism as defined in §2. Next we define

$$D^{i,j} : A_i \xrightarrow{\iota_{i,j}} \text{hom}(A_j^\vee, A_{j-i}^\vee) \xrightarrow{\alpha^{-1}} A_{j-i}^\vee \otimes A_j^{\vee\vee},$$

where $\alpha : \text{hom}(X, Y) \rightarrow Y \otimes X^\vee$ is the canonical morphism. Working dually and employing the reflexivity one also gets

$$D_{i,j} : A_i^\vee \rightarrow A_{j-i} \otimes A_j^\vee.$$

We say that V has *alternating* (resp. *symmetric*) rank $g \in \mathbb{N}_{\geq 1}$ if $L := \wedge^g V$ (resp. $L := \vee^g V$) is invertible and if $\binom{r+i-g}{i}$ (resp. $\binom{r+g-1}{i}$) is invertible in $\text{End}(\mathbb{I})$ for every $0 \leq i \leq g$. Here, for an integer $k \geq 1$,

$$\binom{T}{k} := \frac{1}{k!} T(T-1) \dots (T-k+1) \in \mathbb{Q}[T] \text{ and } \binom{T}{0} = 1.$$

Then we compute, for every $i \leq g$, the compositions

$$\begin{aligned} A_i &\xrightarrow{D^{i,g}} A_{g-i}^\vee \otimes L \xrightarrow{D_{g-i,g} \otimes 1_L} A_i^\vee \otimes L^{-1} \otimes L \simeq A_i^\vee, \\ A_{g-i}^\vee &\xrightarrow{D_{g-i,g}} A_i \otimes L^{-1} \xrightarrow{D^{i,g} \otimes 1_{L^{-1}}} A_{g-i}^\vee \otimes L \otimes L^{-1} \simeq A_{g-i}^\vee \end{aligned}$$

and we prove in Theorem 5.5 (3) (resp. Theorem 6.2 (3)) that, when $A_i = \wedge^i V$ (resp. $A_i = \vee^i V$), they are equal to

$$\begin{aligned} &(-1)^{i(g-i)} \binom{g}{g-i}^{-1} \binom{r-i}{g-i} \text{ (resp. } \binom{g}{g-i}^{-1} \binom{r+g-1}{g-i} \text{)}, \\ &(-1)^{i(g-i)} \binom{g}{i}^{-1} \binom{r+i-g}{i} \text{ (resp. } \binom{g}{i}^{-1} \binom{r+g-1}{i} \text{)}. \end{aligned} \quad (4)$$

In particular, the multiplication maps $\varphi_{i,g-i} : A_i \otimes A_{g-i} \rightarrow A_g$ are perfect pairings for every $0 \leq i \leq g$ (see Corollaries 5.6 and 6.3). We remark that the same constants obtained in (3) and, more generally, for odd degree supervector spaces, matches those in (4) when $r = g$ in the alternating case and, respectively, $r = -g$ in the symmetric case. We say in this case that V has strong alternating or symmetric rank in these cases.

Some remarks are in order about the range of applicability of our results. First of all we note that, in general, the alternating or the symmetric rank may be not uniquely determined. Suppose, however, that we know that there is a field K such that $r \in K \subset \text{End}(\mathbb{I})$ admitting an embedding $\iota : K \hookrightarrow \mathbb{R}$. Then it follows from the formulas $\text{rank}(\wedge^k V) = \binom{r}{k}$ and $\text{rank}(\vee^k V) = \binom{r+k-1}{k}$ (see [AKh, 7.2.4 Proposition] or [De, (7.1.2)]) that we have $r \in \{-1, g\}$ (resp. $r \in \{-g, 1\}$) when V has alternating (resp. symmetric) rank g . In particular, when $r > 0$ (resp. $r < 0$) with respect to the ordering induced by ι , we deduce that $r = g$ (resp. $r = -g$), so that g is a uniquely determined and V has strong alternating (resp. symmetric) rank $g = r$ (resp. $g = -r$).

We recall that V is Kimura positive (resp. negative) when $\wedge^{N+1} V = 0$ (resp. $\vee^{N+1} V = 0$) for $N \geq 0$ large enough. In this case, the formula $\text{rank}(\wedge^k V) = \binom{r}{k}$ (resp. $\text{rank}(\vee^k V) = \binom{r+k-1}{k}$) implies that $r \in \mathbb{Z}_{\geq 0}$ (resp. $r \in \mathbb{Z}_{\leq 0}$) and the smallest integer N such that $\wedge^{N+1} V = 0$ (resp. $\vee^{N+1} V = 0$) is r (resp. $-r$). Furthermore, it is known that in this case, when $\text{End}(\mathbb{I})$ does not have non-trivial idempotents, then $\wedge^r V$ (resp. $\vee^{-r} V$) is invertible (see [Kh, 11.2 Lemma]): in other words V has strong alternating (resp. symmetric) rank $g = r$ (resp. $g = -r$).

In particular, our results applies to the motives $V = h^1(X)$ attached to abelian schemes $X = A$ (see [DM] and [Ku]) or a smooth complete curve $X = C$ over a field (see [Ki1]), which are known to be Kimura negative, while products of an even number of such motives are Kimura positive (see [Ki1] for applications of this notion to the product of two curves). In the subsequent paper [MS] we will apply these results in order to get a motive whose realizations affords two copies of odd weight modular forms on indefinite quaternion algebras. When the quaternion algebra is split, the construction due to Scholl refines and gives a motive whose realizations affords modular forms of both even or odd weight (see [Sc]). Working over an indefinite division quaternion algebra and employing ideas which goes back to [JL], a motive of even weight modular forms has been constructed in [IS] as the kernel of an appropriate Laplace operators. The results of this paper will be used in [MS] in order to show the existence of kernels of Dirac operators which are square-roots of these Laplace operators; the idea of constructing canonical models for the various incarnations of two copies of odd weight modular forms from square roots of the Laplace operators is due, once again, to Jordan and Livné. However, even for these realizations, it is not possible to canonically split them in a single copy: this is possible only including a splitting field for the quaternion algebra in the coefficients, but the resulting splitting depends on the choice of an identification of the base changed algebra with the split quaternion algebra.

Finally, we remark that the perfectness of the multiplication maps gives a Poincaré duality

$$A_i \simeq \text{hom}(A_{g-i}, A_g) \simeq A_g \otimes A_{g-i}^\vee. \quad (5)$$

Indeed, when $V = h^1(A)$ for an abelian scheme A of dimension d , we have that $h^{2d}(A) \simeq \mathbb{I}(-d)$ is invertible and then it is known that

$$\vee^i h^1(A) \simeq h^i(A) \simeq h^{2d-i}(A)^\vee(-d) \simeq h^{2d}(A) \otimes h^{2d-i}(A)^\vee \simeq \vee^{2d} h^1(A) \otimes \vee^{2d-i} h^1(A)^\vee,$$

where the canonical identifications $h^k(A) \simeq \vee^k h^1(A)$ are proved in [Ku, Remarks (3.1.2) (i)], while $h^i(A) \simeq h^{2d-i}(A)^\vee(-d)$ is proved in [DM] (see also [Ku, Remarks (3.1.2) (i)]). This gives a refinement of the motivic Poincaré duality which states that, for a smooth projective scheme X of relative dimension d , we have

$$h(X) \simeq h(X)^\vee(-d). \quad (6)$$

Applying (5) to the motive $h^1(C)$ of a smooth complete curve over a field of genus e , which is Kimura negative of Kimura rank $2e$ with $\vee^{2e} h^1(C) \simeq \mathbb{I}(-e)$ (by [Ki1, Theorem 4.2 and Remark 4.5]), one gets

$$\vee^i h^1(C) \simeq \vee^{2e} h^1(C) \otimes \vee^{2e-i} h^1(C)^\vee \simeq \vee^{2e-i} h^1(C)^\vee(-d)$$

which however, in this case, is not a refinement of (6). We also mention the fact that it is conjectured in [Ki1, Conjecture 7.1] that Chow motives should be Kimura finite, i.e. they should be a direct sum of a Kimura positive and a Kimura negative motive.

The paper is organized as follows. In §2 we develop a general formalism of internal multiplication morphisms attached to a pairing $\varphi : S \otimes X \rightarrow Y$ to be applied to the multiplication morphisms in some algebra object. In §3 we prove the prototype of our Poincaré isomorphism, which only depends on the data of $\varphi_{S,X} : S \otimes X \rightarrow Y$, $\varphi_{X,S} : X \otimes S \rightarrow Y$, $\varphi_{S^\vee, X^\vee} : S^\vee \otimes X^\vee \rightarrow Y^\vee$, $\varphi_{X^\vee, S^\vee} : X^\vee \otimes S^\vee \rightarrow Y^\vee$ subject to an appropriate commutativity constraint and one involving how the internal multiplications are related with respect to the Casimir elements: no associativity constraint is needed for these results. In §4 we apply the above results to the case of algebra objects and include results from §2 in order to get what is the effect of the associativity constraint on internal multiplication morphisms (see Proposition 4.1 and Corollary 4.2). We also make explicit the identifications $\wedge V^\vee \simeq (\wedge V)^\vee$ and $\vee V^\vee \simeq (\vee V)^\vee$ by choosing an appropriate evaluation map as in (1), as useful for the subsequent computations. In §5 we prove the results for the alternating algebras and in §6 we state the results in the symmetric case, the proof being entirely analogous. The key property relating the internal multiplication morphisms with the Casimir elements which is needed to apply the formal Poincaré isomorphism of §3 is proved in §5.1 and the proof requires, besides the two properties of Corollary 4.2, the anti-derivation (resp. derivation) property in case $A = \wedge V$ (resp. $A = \vee V$) which is verified in §5. We also prove various compatibilities of these Poincaré morphisms in Theorem 5.5 and Proposition 5.7 in the alternating case, while the corresponding results in the symmetric case are given in Theorem 6.2 and Proposition 6.4. These further results will be crucial for the applications given in [MS].

2. LINEAR ALGEBRA IN TENSOR CATEGORIES

In the first part of this paper we let \mathcal{C} be an *ACU* additive \otimes -biadditive category with unit object (\mathbb{I}, l, r) and internal homs. We will usually not write the associativity or unitary object constraints explicitly, while the commutativity constraint will be usually denoted by $\tau_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ or by labeling the positions which are switched, such as $\tau_{1,2} \otimes 1_Z = \tau_{X,Y} \otimes 1_Z : X \otimes Y \otimes Z \rightarrow Y \otimes X \otimes Z$ or $\tau_{1,2 \otimes 3} = \tau_{X,Y \otimes Z} : X \otimes Y \otimes Z \rightarrow Y \otimes Z \otimes X$.

To fix notations we recall that the existence of internal homs means that, if $X, Y \in \mathcal{C}$ there is $\text{hom}(X, Y) \in \mathcal{C}$ such that

$$\text{Hom}(S, \text{hom}(X, Y)) = \text{Hom}(S \otimes X, Y) \quad (7)$$

holds as contravariant functors on \mathcal{C} . Taking $S = \text{hom}(X, Y)$ and $1_{\text{hom}(X, Y)}$ yields

$$\text{ev}_{X,Y} : \text{hom}(X, Y) \otimes X \rightarrow Y$$

such that $f : S \rightarrow \text{hom}(X, Y)$ uniquely corresponds to

$$\varphi_f : S \otimes X \xrightarrow{f \otimes 1_X} \text{hom}(X, Y) \otimes X \xrightarrow{\text{ev}_{X,Y}} Y$$

under the identification (7). The opposite evaluation is the composite

$$\text{ev}_{X,Y}^\tau : X \otimes \text{hom}(X, Y) \xrightarrow{\tau_{X, \text{hom}(X, Y)}} \text{hom}(X, Y) \otimes X \xrightarrow{\text{ev}_{X,Y}} Y$$

and $(\text{hom}(X, Y), \text{ev}_{X,Y}^\tau)$ represents $\text{Hom}(X \otimes S, Y)$. Then $(\text{hom}(X, Y), \text{ev}_{X,Y})$, uniquely determined up to a unique isomorphism, is called an internal hom pair for (X, Y) and, when $Y = \mathbb{I}$, we write:

$$(\text{hom}(X, Y), \text{ev}_{X,Y}) = (X^\vee, \text{ev}_X), \quad (\text{hom}(X, Y), \text{ev}_{X,Y}^\tau) = (X^\vee, \text{ev}_X^\tau)$$

and we call (X^\vee, ev_X) a dual pair for X .

We remark that $\text{hom}(X, Y)$ is a bifunctor, contravariant in the first variable and covariant in the second variable as follows. If $f : X_2 \rightarrow X_1$ and $g : Y_1 \rightarrow Y_2$ we define

$$\text{hom}(f, g) : \text{hom}(X_1, Y_1) \rightarrow \text{hom}(X_2, Y_2)$$

as the unique morphism making the following diagram commutative:

$$\begin{array}{ccc} \text{hom}(X_1, Y_1) \otimes X_2 & \xrightarrow{1_{\text{hom}(X_1, Y_1)} \otimes f} & \text{hom}(X_1, Y_1) \otimes X_1 \\ \text{hom}(f, g) \otimes 1_{X_2} \downarrow & & \downarrow g \circ \text{ev}_{X_1, Y_1} \\ \text{hom}(X_2, Y_2) \otimes X_2 & \xrightarrow{\text{ev}_{X_2, Y_2}} & Y_2. \end{array} \quad (8)$$

Note that we have $Hom(1_S, \text{hom}(f, g)) = Hom(1_S \otimes f, g)$ via Yoneda's embedding and (7), from which the functoriality of hom follows.

It follows from this functorial description that hom is biadditive. More explicitly, suppose that we have given biproduct decompositions $X = X^+ \oplus X^-$ and $Y = Y^+ \oplus Y^-$ which are given by injective morphisms $i_X^\pm : X^\pm \rightarrow X$, $i_Y^\pm : Y^\pm \rightarrow Y$, surjective morphisms $p_X^\pm : X \rightarrow X^\pm$, $p_Y^\pm : Y \rightarrow Y^\pm$ and associated idempotents $e_X^\pm : X \rightarrow X$, $e_Y^\pm : Y \rightarrow Y$. The functorial description yields

$$\text{hom}(X, Y) = \text{hom}(X^+, Y^+) \oplus \text{hom}(X^+, Y^-) \oplus \text{hom}(X^-, Y^+) \oplus \text{hom}(X^-, Y^-)$$

associated to the decomposition of $Hom(S \otimes X, Y)$. For $\varepsilon, \eta \in \{\pm\}$, writing $i_{\text{hom}(X^\varepsilon, Y^\eta)} : \text{hom}(X^\varepsilon, Y^\eta) \rightarrow \text{hom}(X, Y)$, $p_{\text{hom}(X^\varepsilon, Y^\eta)} : \text{hom}(X, Y) \rightarrow \text{hom}(X^\varepsilon, Y^\eta)$ and $e_{\text{hom}(X^\varepsilon, Y^\eta)} : \text{hom}(X, Y) \rightarrow \text{hom}(X, Y)$ for the injective and surjective morphisms and the idempotents arising from the decomposition of $Hom(S \otimes X, Y)$ and Yoneda's lemma, one checks

$$i_{\text{hom}(X^\varepsilon, Y^\eta)} = \text{hom}(p_X^\varepsilon, i_Y^\eta), p_{\text{hom}(X^\varepsilon, Y^\eta)} = \text{hom}(i_X^\varepsilon, p_Y^\eta) \text{ and } e_{\text{hom}(X^\varepsilon, Y^\eta)} = \text{hom}(e_X^\varepsilon, e_Y^\eta) \quad (9)$$

as well as

$$\text{ev}_{X^\varepsilon, Y^\eta} = p_Y^\eta \circ \text{ev}_{X, Y} \circ (i_{\text{hom}(X^\varepsilon, Y^\eta)} \otimes i_X^\varepsilon) = p_Y^\eta \circ \text{ev}_{X, Y} \circ (\text{hom}(p_X^\varepsilon, i_Y^\eta) \otimes i_X^\varepsilon). \quad (10)$$

In particular, taking $f : X = X_2 \rightarrow X_1 = Y$ and $g = 1_{\mathbb{I}}$ yields

$$f^\vee := Y^\vee \rightarrow X^\vee$$

and $X \rightsquigarrow X^\vee$ is a contravariant biadditive functor.

We proceed to define standard canonical morphisms. For a totally ordered finite set I , a family $(X_i, Y_i)_{i \in I}$ of objects $X_i, Y_i \in \mathcal{C}$ and a morphism $\varphi : \otimes_{i \in I} Y_i \rightarrow Y$, we may consider

$$\text{ev}_{X_i, Y_i}^{\varphi, I} : (\otimes_{i \in I} \text{hom}(X_i, Y_i) \otimes (\otimes_{i \in I} X_i) \xrightarrow{\tau_{X_i, Y_i}^I} \otimes_{i \in I} (\text{hom}(X_i, Y_i) \otimes X_i) \xrightarrow{\otimes_{i \in I} \text{ev}_{X_i, Y_i}} \otimes_{i \in I} Y_i \xrightarrow{\varphi} Y,$$

where τ_{X_i, Y_i}^I is obtained by appropriately switching the components¹. Then we may define

$$\epsilon_{X_i, Y_i}^{\psi, \varphi, I} : \otimes_{i \in I} \text{hom}(X_i, Y_i) \rightarrow \text{hom}(X, Y)$$

as the unique morphism such that $\text{ev}_{X, Y} \circ (\epsilon_{X_i, Y_i}^{\psi, \varphi, I} \otimes 1_X) = \text{ev}_{X_i, Y_i}^{\psi, \varphi, I}$. When $I = \{1, \dots, i\}$, $X_i = X$ for every i , $Y_i = \mathbb{I}$ for every i and $\varphi : \otimes_{i \in I} \mathbb{I} \xrightarrow{\sim} \mathbb{I}$ is the canonical morphism we write $\tau_X^i := \tau_{X_i, Y_i}^I$, $\text{ev}_X^i := \text{ev}_{X_i, Y_i}^{\varphi, I}$ and $\epsilon_{X_i, Y_i}^{\varphi, I} := \epsilon_X^i$.

The morphisms

$$i_X : X \rightarrow X^{\vee\vee} \text{ and } \alpha_{X, Y} : Y \otimes X^\vee \rightarrow \text{hom}(X, Y)$$

are defined, respectively, as the unique morphisms making the following diagrams commutative:

$$\begin{array}{ccc} X \otimes X^\vee & & Y \otimes X^\vee \otimes X \\ \downarrow i_X \otimes 1_{X^\vee} & \searrow \text{ev}_X^\tau & \downarrow \alpha_{X, Y} \otimes 1_X \\ X^{\vee\vee} \otimes X^\vee & \xrightarrow{\text{ev}_{X^\vee}} & \mathbb{I} \end{array} \quad \begin{array}{ccc} Y \otimes X^\vee \otimes X & & \\ \downarrow \alpha_{X, Y} \otimes 1_X & \searrow 1_Y \otimes \text{ev}_X & \\ \text{hom}(X, Y) \otimes X & \xrightarrow{\text{ev}_{X, Y}} & Y. \end{array}$$

We may consider the morphism

$$\text{hom}(Y, Z) \otimes \text{hom}(X, Y) \otimes X \xrightarrow{1_{\text{hom}(Y, Z)} \otimes \text{ev}_{X, Y}} \text{hom}(Y, Z) \otimes Y \xrightarrow{\text{ev}_{Y, Z}} Z$$

and define the internal composition law

$$c = c_{X, Y, Z} = (\cdot) \circ (\cdot) : \text{hom}(Y, Z) \otimes \text{hom}(X, Y) \rightarrow \text{hom}(X, Z)$$

as the unique morphism such that

$$\text{ev}_{X, Z} \circ (c_{X, Y, Z} \otimes 1_X) = \text{ev}_{Y, Z} \circ (1_{\text{hom}(Y, Z)} \otimes \text{ev}_{X, Y}).$$

¹In symbols,

$$\tau_{X_i, Y_i}^I ((\otimes_{i \in I} f_i) \otimes (\otimes_{i \in I} x_i)) := \otimes_{i \in I} (f_i \otimes x_i).$$

The opposite internal composition law is defined as the composite

$$c_{X,Y,Z}^\tau : \text{hom}(X, Y) \otimes \text{hom}(Y, Z) \xrightarrow{\tau_{\text{hom}(X,Y), \text{hom}(Y,Z)}} \text{hom}(Y, Z) \otimes \text{hom}(X, Y) \xrightarrow{c_{X,Y,Z}} \text{hom}(X, Z)$$

The following result is easily established.

Lemma 2.1. *Suppose that we have given*

$$f : S \rightarrow \text{hom}(X, Y) \text{ and } g : T \rightarrow \text{hom}(Y, Z)$$

which correspond, under (γ) , to morphisms

$$\varphi_f : S \otimes X \rightarrow Y \text{ and } \varphi_g : T \otimes Y \rightarrow Z.$$

Then

$$c_{X,Y,Z} \circ (g \otimes f) : T \otimes S \xrightarrow{g \otimes f} \text{hom}(Y, Z) \otimes \text{hom}(X, Y) \xrightarrow{c_{X,Y,Z}} \mathbf{Hom}(X, Z)$$

corresponds, under (γ) , to the morphism

$$\varphi_g \circ (1_T \otimes \varphi_f) : T \otimes S \otimes X \rightarrow Z.$$

In addition to the "external" duality morphism $\text{Hom}(X, Y) \rightarrow \text{Hom}(Y^\vee, X^\vee)$, the category \mathcal{C} is endowed with an internal duality morphism

$$d_{X,Y} : \text{hom}(X, Y) \rightarrow \text{hom}(Y^\vee, X^\vee),$$

which is by definition the unique morphism making the following diagram commutative:

$$\begin{array}{ccc} \text{hom}(X, Y) \otimes Y^{\tau_{\text{hom}(X,Y), Y^\vee}} \otimes \text{hom}(X, Y) & & \\ d_{X,Y} \otimes 1_{Y^\vee} \downarrow & & \downarrow c_{X,Y,I} \\ \text{hom}(Y^\vee, X^\vee) \otimes Y^\vee & \xrightarrow{\text{ev}_{Y^\vee, X^\vee}} & X^\vee. \end{array}$$

It enjoys a number of expected properties, namely it makes commutative the following diagrams.

- It is the unique morphism making the following diagram commutative²:

$$\begin{array}{ccc} \text{hom}(X, Y) \otimes X \otimes Y^\vee & \xrightarrow{(1_X \otimes d_{X,Y} \otimes 1_{Y^\vee}) \circ (\tau_{\text{hom}(X,Y), X \otimes 1_{Y^\vee}})} & X \otimes \text{hom}(Y^\vee, X^\vee) \otimes Y^\vee \\ \text{ev}_{X,Y} \otimes 1_{Y^\vee} \downarrow & & \downarrow 1_X \otimes \text{ev}_{Y^\vee, X^\vee} \\ Y \otimes Y^\vee & \xrightarrow{\text{ev}_Y^\tau} \mathbb{I} \xleftarrow{\text{ev}_X^\tau} & X \otimes X^\vee \end{array} \quad (11)$$

- The following diagram is commutative³:

$$\begin{array}{ccc} \text{hom}(Y, Z) \otimes \text{hom}(X, Y) & \xrightarrow{d_{Y,Z} \otimes d_{X,Y}} & \text{hom}(Z^\vee, Y^\vee) \otimes \text{hom}(Y^\vee, X^\vee) \\ c_{X,Y,Z} \downarrow & & \downarrow c_{Z^\vee, Y^\vee, X^\vee}^\tau \\ \text{hom}(X, Z) & \xrightarrow{d_{X,Z}} & \text{hom}(Z^\vee, X^\vee). \end{array} \quad (12)$$

²In symbols, setting $f^\vee := d_{X,Y}(f)$ for $f \in \text{hom}(X, Y)$,

$\langle f(x), y^\vee \rangle = \langle x, f^\vee(y^\vee) \rangle$ for $x \in X$ and $y^\vee \in Y^\vee$.

³In symbols, for $f \in \text{hom}(X, Y)$ and $g \in \text{hom}(Y, Z)$,

$(g \circ f)^\vee = g^\vee \circ^{opp} f^\vee = f^\vee \circ g^\vee$.

- If we have given $f : X_2 \rightarrow X_1$ and $g : Y_1 \rightarrow Y_2$ the following diagram is commutative:

$$\begin{array}{ccc} \text{hom}(X_1, Y_1) & \xrightarrow{d_{X_1, Y_1}} & \text{hom}(Y_1^\vee, X_1^\vee) \\ \text{hom}(f, g) \downarrow & & \downarrow \text{hom}(g^\vee, f^\vee) \\ \text{hom}(X_2, Y_2) & \xrightarrow{d_{X_2, Y_2}} & \text{hom}(Y_2^\vee, X_2^\vee). \end{array} \quad (13)$$

- The following further diagrams are commutative⁴:

$$\begin{array}{ccc} Y \otimes X^\vee & \xrightarrow{\alpha_{X, Y}} & \text{hom}(X, Y) \\ \downarrow (1_{X^\vee} \otimes i_Y) \circ \tau_{Y, X^\vee} & & \downarrow d_{X, Y} \\ X^\vee \otimes Y^{\vee\vee} & \xrightarrow{\alpha_{Y^\vee, X^\vee}} & \text{hom}(Y^\vee, X^\vee) \end{array} \quad \begin{array}{ccc} \text{hom}(X, Y) & \xrightarrow{\text{hom}(1_X, i_Y)} & \text{hom}(X, Y^{\vee\vee}) \\ \downarrow d_{Y^\vee, X^\vee} \circ d_{X, Y} & & \downarrow \text{hom}(i_X, 1_{Y^{\vee\vee}}) \\ \text{hom}(X^{\vee\vee}, Y^{\vee\vee}) & \xrightarrow{\text{hom}(i_X, 1_{Y^{\vee\vee}})} & \text{hom}(X, Y^{\vee\vee}) \end{array} \quad (14)$$

We recall that \mathcal{C} is rigid whenever the morphisms ϵ_{X_i, Y_i}^I and i_X are isomorphisms and is said to be pseudo-abelian when idempotents have kernels (and then also cokernels).

We will employ the following notation: a label (\otimes) (resp. (τ)) placed in the middle of a diagram will mean that the diagram is commutative by functoriality of \otimes (resp. the τ constraint).

2.1. Abstract internal multiplication. Suppose that we have given a morphism

$$f : S \rightarrow \text{hom}(X, Y) \text{ corresponding to } \varphi_f : S \otimes X \xrightarrow{f \otimes 1_X} \text{hom}(X, Y) \otimes X \xrightarrow{\text{ev}_{X, Y}} Y.$$

Then we define the corresponding "internal multiplication" morphism as the composite:

$$\iota_f : S \xrightarrow{f} \text{hom}(X, Y) \xrightarrow{d_{X, Y}} \text{hom}(Y^\vee, X^\vee) \text{ corresponding to } \varphi_{\iota_f} : S \otimes Y^\vee \xrightarrow{\iota_f \otimes 1_{Y^\vee}} \text{hom}(Y^\vee, X^\vee) \otimes Y^\vee \xrightarrow{\text{ev}_{Y^\vee, X^\vee}} X^\vee.$$

One checks that (11) implies that the following diagram is commutative:

$$\begin{array}{ccc} S \otimes X \otimes Y^\vee & \xrightarrow{(1_X \otimes \varphi_{\iota_f}) \circ (\tau_{S, X} \otimes 1_{Y^\vee})} & X \otimes X^\vee \\ \downarrow \varphi_f \otimes 1_{Y^\vee} & & \downarrow \text{ev}_X^\tau \\ Y \otimes Y^\vee & \xrightarrow{\text{ev}_Y^\tau} & \mathbb{I}. \end{array} \quad (15)$$

Remark 2.2. The morphism φ_{ι_f} , and hence ι_f , is characterized by the property of making (15) commutative.

Suppose now that we have also given:

$$\begin{aligned} g : T &\rightarrow \text{hom}(Y, Z) \text{ corresponding to } \varphi_g : T \otimes Y \rightarrow Z, \\ h : U &\rightarrow \text{hom}(X, Z) \text{ corresponding to } \varphi_h : U \otimes X \rightarrow Z, \\ k : T &\rightarrow \text{hom}(S, U) \text{ corresponding to } \varphi_k : T \otimes S \rightarrow U. \end{aligned}$$

As an application of Lemma 2.1, we have the equivalence:

$$\begin{array}{ccc} T \otimes S \otimes X & \xrightarrow{1_T \otimes \varphi_f} & T \otimes Y \\ \downarrow \varphi_k \otimes 1_X & \circlearrowleft & \downarrow \varphi_g \\ U \otimes X & \xrightarrow{\varphi_h} & Z \end{array} \Leftrightarrow \begin{array}{ccc} T \otimes S & \xrightarrow{g \otimes f} & \text{hom}(Y, Z) \otimes \text{hom}(X, Y) \\ \downarrow \varphi_k & \circlearrowleft & \downarrow c_{X, Y, Z} \\ U & \xrightarrow{h} & \text{hom}(X, Z) \end{array} \quad (16)$$

We also have the associated internal multiplication morphisms:

$$\begin{aligned} \iota_g & : T \xrightarrow{g} \text{hom}(Y, Z) \xrightarrow{d_{Y, Z}} \text{hom}(Z^\vee, Y^\vee) \text{ corresponding to } \varphi_{\iota_g} : T \otimes Z^\vee \rightarrow Y^\vee, \\ \iota_h & : U \xrightarrow{h} \text{hom}(X, Z) \xrightarrow{d_{X, Z}} \text{hom}(Z^\vee, X^\vee) \text{ corresponding to } \varphi_{\iota_h} : U \otimes Z^\vee \rightarrow X^\vee. \end{aligned}$$

⁴In symbols, the second commutative diagram tells that $f^{\vee\vee} = f$ up to the identification $X^{\vee\vee} = X$ and $Y^{\vee\vee} = Y$ whenever X and Y are reflexive.

Consider the morphism:

$$\varphi_k^\tau : S \otimes T \xrightarrow{\tau_{S,T}} T \otimes S \xrightarrow{\varphi_k} U.$$

The equivalence (16), applied with (f, g, h, φ_k) replaced by $(\iota_g, \iota_h, \iota_f, \varphi_k^\tau)$, easily translates into the equivalence:

$$\begin{array}{ccc} S \otimes T \otimes Z^\vee & \xrightarrow{1_S \otimes \varphi_{\iota_g}} & S \otimes Y^\vee \\ \varphi_k^\tau \otimes 1_{Z^\vee} \downarrow & \circlearrowleft & \downarrow \varphi_{\iota_f} \\ U \otimes Z^\vee & \xrightarrow{\varphi_{\iota_h}} & X^\vee \end{array} \Leftrightarrow \begin{array}{ccc} T \otimes S & \xrightarrow{\iota_g \otimes \iota_f} & \text{hom}(Z^\vee, Y^\vee) \otimes \text{hom}(Y^\vee, X^\vee) \\ \varphi_k \downarrow & \circlearrowleft & \downarrow c_{Z^\vee, Y^\vee, X^\vee}^\tau \\ U & \xrightarrow{\iota_h} & \text{hom}(Z^\vee, X^\vee). \end{array} \quad (17)$$

Finally we remark that the second square of the following diagram is commutative by (12):

$$\begin{array}{ccccc} T \otimes S & \xrightarrow{g \otimes f} & \text{hom}(Y, Z) \otimes \text{hom}(X, Y) & \xrightarrow{d_{Y,Z} \otimes d_{X,Y}} & \text{hom}(Z^\vee, Y^\vee) \otimes \text{hom}(Y^\vee, X^\vee) \\ \varphi_k \downarrow & & \downarrow c_{X,Y,Z} & & \downarrow c_{Z^\vee, Y^\vee, X^\vee}^\tau \\ U & \xrightarrow{h} & \text{hom}(X, Z) & \xrightarrow{d_{X,Z}} & \text{hom}(Z^\vee, X^\vee). \end{array}$$

It follows that we have the implication

$$(16) \text{ commutative} \Rightarrow (17) \text{ commutative}^5 \quad (18)$$

We now turn to the consideration of how the formation of internal multiplication behaves with respect to biproduct decompositions. To this end we assume that, for all the objects W considered above, we have given a biproduct decomposition $W = W^+ \oplus W^-$ obtained by means of injective morphisms $i_W^\pm : W^\pm \rightarrow W$, surjective morphisms $p_W^\pm : W \rightarrow W^\pm$ and associated idempotents $e_W^\pm : W \rightarrow W$.

For $(\varepsilon_1, \eta_1, \nu_1), (\varepsilon_2, \eta_2, \nu_2) \in \{\pm\} \times \{\pm\} \times \{\pm\}$, define the following morphisms:

$$\begin{aligned} f^{\varepsilon_1, \varepsilon_2; \eta_2} : S^{\varepsilon_1} &\xrightarrow{i_S^{\varepsilon_1}} S \xrightarrow{f} \text{hom}(X, Y) \xrightarrow{p_{\text{hom}(X^{\varepsilon_2}, Y^{\eta_2})}} \text{hom}(X^{\varepsilon_2}, Y^{\eta_2}), \\ \iota_f^{\varepsilon_1, \eta_2; \varepsilon_2} : S^{\varepsilon_1} &\xrightarrow{i_S^{\varepsilon_1}} S \xrightarrow{\iota_f} \text{hom}(Y^\vee, X^\vee) \xrightarrow{p_{\text{hom}(Y^{\eta_2^\vee}, X^{\varepsilon_2^\vee})}} \text{hom}(Y^{\eta_2^\vee}, X^{\varepsilon_2^\vee}), \\ \varphi_f^{\varepsilon_1, \varepsilon_2; \eta_2} : S^{\varepsilon_1} \otimes X^{\varepsilon_2} &\xrightarrow{i_S^{\varepsilon_1} \otimes i_X^{\varepsilon_2}} S \otimes X \xrightarrow{\varphi_f} Y \xrightarrow{p_Y^{\eta_2}} Y^{\eta_2}, \\ \varphi_{\iota_f}^{\varepsilon_1, \eta_2; \varepsilon_2} : S^{\varepsilon_1} \otimes Y^{\eta_2^\vee} &\xrightarrow{i_S^{\varepsilon_1} \otimes i_{Y^\vee}^{\eta_2}} S \otimes Y^\vee \xrightarrow{\varphi_{\iota_f}} X^\vee \xrightarrow{p_{X^\vee}^{\varepsilon_2}} X^{\varepsilon_2^\vee} \end{aligned}$$

as well as the morphisms $g^{\eta_1, \eta_2; \nu_2}, h^{\nu_1, \varepsilon_2; \nu_2}, k^{\eta_1, \varepsilon_1; \nu_1}$ and the other defined similarly as for f .

Lemma 2.3. *Writing $\varphi_{f^{\varepsilon_1, \varepsilon_2; \eta_2}} : S^{\varepsilon_1} \otimes X^{\varepsilon_2} \rightarrow Y^{\eta_2}$ (resp. $\varphi_{\iota_f^{\varepsilon_1, \eta_2; \varepsilon_2}} : S^{\varepsilon_1} \otimes Y^{\eta_2^\vee} \rightarrow X^{\varepsilon_2^\vee}$) for the morphism corresponding to $f^{\varepsilon_1, \varepsilon_2; \eta_2} : S^{\varepsilon_1} \rightarrow \text{hom}(X^{\varepsilon_2}, Y^{\eta_2})$ (resp. $\iota_f^{\varepsilon_1, \eta_2; \varepsilon_2} : S^{\varepsilon_1} \rightarrow \text{hom}(Y^{\eta_2^\vee}, X^{\varepsilon_2^\vee})$), we have $\varphi_{f^{\varepsilon_1, \varepsilon_2; \eta_2}} = \varphi_f^{\varepsilon_1, \varepsilon_2; \eta_2}$ and $\varphi_{\iota_f^{\varepsilon_1, \eta_2; \varepsilon_2}} = \varphi_{\iota_f}^{\varepsilon_1, \eta_2; \varepsilon_2}$ as well as $\iota_{f^{\varepsilon_1, \varepsilon_2; \eta_2}} = \iota_f^{\varepsilon_1, \eta_2; \varepsilon_2}$ or, equivalently, $\varphi_{\iota_{f^{\varepsilon_1, \varepsilon_2; \eta_2}}} = \varphi_{\iota_f^{\varepsilon_1, \eta_2; \varepsilon_2}} = \varphi_{\iota_f}^{\varepsilon_1, \eta_2; \varepsilon_2}$.*

Proof. The equality $\varphi_{f^{\varepsilon_1, \varepsilon_2; \eta_2}} = \varphi_f^{\varepsilon_1, \varepsilon_2; \eta_2}$ is a consequence of $p_{\text{hom}(X^{\varepsilon_2}, Y^{\eta_2})} = \text{hom}(i_X^{\varepsilon_2}, p_Y^{\eta_2})$ given by (9) and the characterizing property (8) and $\varphi_{\iota_f^{\varepsilon_1, \eta_2; \varepsilon_2}} = \varphi_{\iota_f}^{\varepsilon_1, \eta_2; \varepsilon_2}$ is proved in the same way. Next, consider the following diagram:

$$\begin{array}{ccccc} S^{\varepsilon_1} & \xrightarrow{i_S^{\varepsilon_1}} & S & \xrightarrow{f} & \text{hom}(X, Y) \xrightarrow{d_{X,Y}} \text{hom}(Y^\vee, X^\vee) \\ & & & & \downarrow p_{\text{hom}(X^{\varepsilon_2}, Y^{\eta_2})} \\ & & & & \text{hom}(X^{\varepsilon_2}, Y^{\eta_2}) \xrightarrow{d_{X^{\varepsilon_2}, Y^{\eta_2}}} \text{hom}(Y^{\eta_2^\vee}, X^{\varepsilon_2^\vee}). \\ & & & & \downarrow p_{\text{hom}(Y^{\eta_2^\vee}, X^{\varepsilon_2^\vee})} \end{array}$$

The square is commutative because

$$\text{hom}(i_X^{\varepsilon_2}, p_Y^{\eta_2}) = p_{\text{hom}(X^{\varepsilon_2}, Y^{\eta_2})}$$

and

$$\text{hom}\left((p_Y^{\eta_2})^\vee, (i_X^{\varepsilon_2})^\vee\right) = \text{hom}(i_{Y^\vee}^{\eta_2}, p_{X^\vee}^{\varepsilon_2}) = p_{\text{hom}(Y^{\eta_2^\vee}, X^{\varepsilon_2^\vee})},$$

again by (9) and because the duality is a contravariant and additive functor, so that we may apply (13). But we have

$$\begin{aligned} p_{\text{hom}(Y^{\eta_2 \vee}, X^{\varepsilon_2 \vee})} \circ d_{X,Y} \circ f \circ i_S^{\varepsilon_1} &= p_{\text{hom}(Y^{\eta_2 \vee}, X^{\varepsilon_2 \vee})} \circ \iota_f \circ i_S^{\varepsilon_1} = \iota_f^{\varepsilon_1, \eta_2; \varepsilon_2}, \\ d_{X^{\varepsilon_2}, Y^{\eta_2}} \circ p_{\text{hom}(X^{\varepsilon_2}, Y^{\eta_2})} \circ f \circ i_S^{\varepsilon_1} &= d_{X^{\varepsilon_2}, Y^{\eta_2}} \circ f^{\varepsilon_1, \varepsilon_2; \eta_2} = \iota_{f^{\varepsilon_1, \varepsilon_2; \eta_2}}. \end{aligned}$$

□

It follows from Lemma 2.3 (also applied to g , h and k) that we may apply the above considerations with (f, g, h, φ_k) replaced by $(f^{\varepsilon_1, \varepsilon_2; \eta_2}, g^{\eta_1, \eta_2; \nu_2}, h^{\nu_1, \varepsilon_2; \nu_2}, \varphi_k^{\eta_1, \varepsilon_1; \nu_1})$. Hence, we deduce that

$$\begin{array}{ccc} S^{\varepsilon_1} \otimes X^{\varepsilon_2} \otimes Y^{\eta_2 \vee} & \xrightarrow{(1_{X^{\varepsilon_2}} \otimes \varphi_{\iota_f}^{\varepsilon_1, \eta_2; \varepsilon_2}) \circ (\tau_{S^{\varepsilon_1}, X^{\varepsilon_2}} \otimes 1_{Y^{\eta_2 \vee}})} & X^{\varepsilon_2} \otimes X^{\varepsilon_2 \vee} \\ \downarrow \varphi_f^{\varepsilon_1, \varepsilon_2; \eta_2} \otimes 1_{Y^{\eta_2 \vee}} & & \downarrow \text{ev}_{X^{\varepsilon_2}}^\tau \\ Y^{\eta_2} \otimes Y^{\eta_2 \vee} & \xrightarrow{\text{ev}_{Y^{\eta_2}}^\tau} & \mathbb{I} \end{array} \quad (19)$$

is commutative and that we have the implications:

$$\begin{array}{ccc} T^{\eta_1} \otimes S^{\varepsilon_1} \otimes X^{\varepsilon_2} & \xrightarrow{1_{T^{\eta_1}} \otimes \varphi_f^{\varepsilon_1, \varepsilon_2; \eta_2}} & T^{\eta_1} \otimes Y^{\eta_2} \\ \downarrow \varphi_k^{\eta_1, \varepsilon_1; \nu_1} \otimes 1_{X^{\varepsilon_2}} & \circlearrowleft & \downarrow \varphi_g^{\eta_1, \eta_2; \nu_2} \\ U^{\nu_1} \otimes X^{\varepsilon_2} & \xrightarrow{\varphi_h^{\nu_1, \varepsilon_2; \nu_2}} & Z^{\nu_2} \end{array} \Leftrightarrow \begin{array}{ccc} T^{\eta_1} \otimes S^{\varepsilon_1} & \xrightarrow{g^{\eta_1, \eta_2; \nu_2} \otimes f^{\varepsilon_1, \varepsilon_2; \eta_2}} & \text{hom}(Y^{\eta_2}, Z^{\nu_2}) \otimes \text{hom}(X^{\varepsilon_2}, Y^{\eta_2}) \\ \downarrow \varphi_k^{\eta_1, \varepsilon_1; \nu_1} & \circlearrowleft & \downarrow c_{X^{\varepsilon_2}, Y^{\eta_2}, Z^{\nu_2}} \\ U^{\nu_1} & \xrightarrow{h^{\nu_1, \varepsilon_2; \nu_2}} & \text{hom}(X^{\varepsilon_2}, Z^{\nu_2}), \end{array} \quad (20)$$

$$\begin{array}{ccc} S^{\varepsilon_1} \otimes T^{\eta_1} \otimes Z^{\nu_2 \vee} & \xrightarrow{1_{S^{\varepsilon_1}} \otimes \varphi_{\iota_g}^{\eta_1, \nu_2; \eta_2}} & S^{\varepsilon_1} \otimes Y^{\eta_2 \vee} \\ \downarrow \varphi_k^{\eta_1, \varepsilon_1; \nu_1} \otimes 1_{Z^{\nu_2 \vee}} & \circlearrowleft & \downarrow \varphi_{\iota_f}^{\varepsilon_1, \eta_2; \varepsilon_2} \\ U^{\nu_1} \otimes Z^{\nu_2 \vee} & \xrightarrow{\varphi_{\iota_h}^{\nu_1, \nu_2; \varepsilon_2}} & X^{\varepsilon_2 \vee} \end{array} \Leftrightarrow \begin{array}{ccc} T^{\eta_1} \otimes S^{\varepsilon_1} & \xrightarrow{g^{\eta_1, \nu_2; \eta_2} \otimes \varphi_f^{\varepsilon_1, \eta_2; \varepsilon_2}} & \text{hom}(Z^{\nu_2 \vee}, Y^{\eta_2 \vee}) \otimes \text{hom}(Y^{\eta_2 \vee}, X^{\varepsilon_2 \vee}) \\ \downarrow \varphi_k^{\eta_1, \varepsilon_1; \nu_1} & \circlearrowleft & \downarrow c_{Z^{\nu_2 \vee}, Y^{\eta_2 \vee}, X^{\varepsilon_2 \vee}}^\tau \\ U^{\nu_1} & \xrightarrow{\iota_h^{\nu_1, \nu_2; \varepsilon_2}} & \text{hom}(Z^{\nu_2 \vee}, X^{\varepsilon_2 \vee}), \end{array} \quad (21)$$

as well as

$$(20) \text{ commutative} \Rightarrow (21) \text{ commutative}. \quad (22)$$

The proof of the following Lemma is just a formal computation.

Lemma 2.4. *Suppose that $e_Z^{\nu_2} \circ \varphi_h \circ (e_U^{\nu_1} \otimes 1_X) = e_Z^{\nu_2} \circ \varphi_h$ and $e_Z^{\nu_2} \circ \varphi_g \circ (1_T \otimes e_Y^{\eta_2}) = e_Z^{\nu_2} \circ \varphi_g$. Then we have the implication (16) commutative \Rightarrow (20) commutative.*

The following result is now a combination of the commutativity of (19), Lemma 2.4 and (22).

Proposition 2.5. *Suppose that $e_Z^{\nu_2} \circ \varphi_h \circ (e_U^{\nu_1} \otimes 1_X) = e_Z^{\nu_2} \circ \varphi_h$ and $e_Z^{\nu_2} \circ \varphi_g \circ (1_T \otimes e_Y^{\eta_2}) = e_Z^{\nu_2} \circ \varphi_g$ and that (16) is commutative. Then (19), (20) and (21) are commutative.*

For future reference it will be convenient to introduce some more notation. When $\alpha_{X,Y}$ is an isomorphism, we define

$$D_f : S \xrightarrow{f} \text{hom}(X, Y) \xrightarrow{\alpha_{X,Y}^{-1}} Y \otimes X^\vee.$$

Suppose now that we have given $g : S \rightarrow \text{hom}(X^\vee, Y^\vee)$, so that we have $\iota_g : S \rightarrow \text{hom}(Y^{\vee \vee}, X^{\vee \vee})$. When X is reflexive, we define

$$\iota_g^* : S \xrightarrow{\iota_g} \text{hom}(Y^{\vee \vee}, X^{\vee \vee}) \xrightarrow{\text{hom}(i_Y, i_X^{-1})} \text{hom}(Y, X).$$

Suppose that X is reflexive and that both α_{X^\vee, Y^\vee} and $\alpha_{Y, X}$ are isomorphisms. Then it follows from the first commutative diagram of (14) that $d_{Y, X}$ is an isomorphism and then, from the second commutative diagram of (14), we deduce that $\text{hom}(i_Y, i_X^{-1}) \circ d_{X^\vee, Y^\vee} = d_{Y, X}^{-1}$, so that

$$\begin{aligned} \iota_{\iota_g^*} & : &= d_{Y, X} \circ \iota_g^* &= d_{Y, X} \circ \text{hom}(i_Y, i_X^{-1}) \circ d_{X^\vee, Y^\vee} \circ g \\ &= &d_{Y, X} \circ d_{Y, X}^{-1} \circ g &= g. \end{aligned}$$

It follows from (15) that the first of the subsequently commutative diagrams commutes:

$$\begin{array}{ccc}
S \otimes Y \otimes X^\vee & \xrightarrow{(1_Y \otimes \varphi_g) \circ (\tau_{S,Y} \otimes 1_{X^\vee})} & Y \otimes Y^\vee \\
\downarrow \varphi_{i_g^*} \otimes 1_{X^\vee} & & \downarrow \text{ev}_Y^\tau \\
X \otimes X^\vee & \xrightarrow{\text{ev}_X^\tau} & \mathbb{I}
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{ccc}
S \otimes X^\vee \otimes Y & \xrightarrow{(1_{X^\vee} \otimes \varphi_{i_g^*}) \circ (\tau_{S,X^\vee} \otimes 1_Y)} & X^\vee \otimes X \\
\downarrow \varphi_g \otimes 1_Y & & \downarrow \text{ev}_X \\
Y^\vee \otimes Y & \xrightarrow{\text{ev}_Y} & \mathbb{I}.
\end{array}
\quad (23)$$

Here the equivalence is easily obtained by applying $1_S \otimes \tau_{X^\vee,Y}$ (resp. $1_S \otimes \tau_{Y,X^\vee}$) to the first (resp. second) diagram to get the second (resp. first) diagram. Also, it follows from the functorial description $\text{Hom}(1_S, \text{hom}(i_Y, i_X^{-1})) = \text{Hom}(1_S \otimes i_Y, i_X^{-1})$ (up to (7)) of hom that the following diagram is commutative:

$$\begin{array}{ccc}
S \otimes Y & \xrightarrow{\varphi_{i_g^*}} & X \\
\downarrow 1_S \otimes i_Y & & \downarrow i_X \\
S \otimes Y^{\vee\vee} & \xrightarrow{\varphi_{i_g}} & X^{\vee\vee}.
\end{array}
\quad (24)$$

Finally, in addition to $D_{i_g} : S \rightarrow X^{\vee\vee} \otimes Y^{\vee\vee\vee}$, defined when $\alpha_{Y^{\vee\vee}, X^{\vee\vee}}$ is an isomorphism, we may define, when X is reflexive and $\alpha_{Y,X}$ is an isomorphism:

$$D_{i_g^*} : S \xrightarrow{i_g^*} \text{hom}(Y, X) \xrightarrow{\alpha_{Y,X}^{-1}} X \otimes Y^\vee.$$

The relationship between D_{i_g} and $D_{i_g^*}$ can be made explicit as follows. Consider the following diagram:

$$\begin{array}{ccccc}
S & \xrightarrow{g} & \text{hom}(X^\vee, Y^\vee) & \xrightarrow{d_{X^\vee, Y^\vee}} & \text{hom}(Y^{\vee\vee}, X^{\vee\vee}) & \xrightarrow{\text{hom}(i_Y, i_X^{-1})} & \text{hom}(Y, X) \\
& & \uparrow \alpha_{X^\vee, Y^\vee} & & \uparrow \alpha_{Y^{\vee\vee}, X^{\vee\vee}} & & \uparrow \alpha_{Y,X} \\
& & Y^\vee \otimes X^{\vee\vee\vee} & \xrightarrow{(1_{X^{\vee\vee\vee}} \otimes i_{Y^\vee}) \circ \tau_{Y^\vee, X^{\vee\vee\vee}}} & X^{\vee\vee\vee} \otimes Y^{\vee\vee\vee} & \xrightarrow{i_X^{-1} \otimes (i_Y)^\vee} & X \otimes Y^\vee.
\end{array}
\quad (25)$$

The first square is commutative thanks to the first diagram in (14), while the second square is commutative by functoriality of α . The subsequent lemma, whose proof we leave to the reader, shows that, when Y is reflexive, $(i_Y)^\vee = i_{Y^\vee}^{-1}$ and we find, in this case,

$$D_{i_g} = (i_X \otimes i_{Y^\vee}) \circ D_{i_g^*} \quad (26)$$

Lemma 2.6. *We have the equality $(i_X)^\vee \circ i_{X^\vee} = 1_{X^\vee}$. In particular, if X is reflexive, then X^\vee is reflexive and $i_{X^\vee}^{-1} = (i_X)^\vee$.*

The following lemma will be useful later.

Lemma 2.7. *Suppose that we have given $f : S \rightarrow \text{hom}(X, Y)$ and, respectively, $g : S \rightarrow \text{hom}(X^\vee, Y^\vee)$, that α_{Y^\vee, X^\vee} is an isomorphism, so that D_{i_f} is defined, and, respectively, that $\alpha_{Y^{\vee\vee}, X^{\vee\vee}}$ is an isomorphism and that X is reflexive and $\alpha_{Y,X}$ is an isomorphism, so that $D_{i_g^*}$ is defined. Then the first and, respectively, the second of the following diagrams is commutative:*

$$\begin{array}{ccc}
S \otimes X & \xrightarrow{\varphi_f} & Y \\
\downarrow D_{i_f} \otimes 1_X & & \downarrow i_Y \\
X^\vee \otimes Y^{\vee\vee} \otimes X & \xrightarrow{\text{ev}_{13, Y^{\vee\vee}}^\phi} & Y^{\vee\vee}
\end{array}
\quad
\begin{array}{ccc}
S \otimes X^\vee & \xrightarrow{\varphi_g} & Y^\vee \\
\downarrow D_{i_g^*} \otimes 1_{X^\vee} & & \downarrow \text{ev}_{13, Y^\vee}^\tau \\
X \otimes Y^\vee \otimes X^\vee & \xrightarrow{\text{ev}_{13, Y^\vee}^\tau} & Y^\vee
\end{array}$$

where $\text{ev}_{13, Y^{\vee\vee}}^\phi := (1_{Y^{\vee\vee}} \otimes \text{ev}_X) \circ (\tau_{X^\vee, Y^{\vee\vee}} \otimes 1_X)$ and $\text{ev}_{13, Y^\vee}^\tau := (1_{Y^\vee} \otimes \text{ev}_X) \circ (\tau_{X, Y^\vee} \otimes 1_{X^\vee})$.

Proof. Consider the following diagram, where we set $t_{Y,X} := (1_{X^\vee} \otimes i_Y) \circ \tau_{Y,X^\vee}$:

$$\begin{array}{ccccc}
& & Y & \xrightarrow{i_Y} & Y^{\vee\vee} \\
& \nearrow \text{ev}_{X,Y} & \uparrow 1_Y \otimes \text{ev}_X & (\otimes) & \uparrow 1_{Y^{\vee\vee}} \otimes \text{ev}_X \\
\text{hom}(X,Y) \otimes X & \xleftarrow{\alpha_{X,Y} \otimes 1_X} & Y \otimes X^\vee \otimes X & \xrightarrow{i_Y \otimes 1_{X^\vee} \otimes X} & Y^{\vee\vee} \otimes X^\vee \otimes X \\
\downarrow d_{X,Y} \otimes 1_X & (B) & \downarrow t_{Y,X} \otimes 1_X & (\tau) & \downarrow \tau_{Y^{\vee\vee},X^\vee} \otimes 1_X \\
\text{hom}(Y^\vee, X^\vee) \otimes X & \xleftarrow{\alpha_{Y^\vee, X^\vee} \otimes 1_{X^\vee}} & X^\vee \otimes Y^{\vee\vee} \otimes X & &
\end{array}$$

The region (A) is commutative by definition of $\alpha_{X,Y}$ and (B) thanks to the first diagram in (14). Noticing that $\tau_{X^\vee, Y^{\vee\vee}} = (\tau_{Y^{\vee\vee}, X^\vee})^{-1}$ we deduce

$$\begin{aligned}
i_Y \circ \varphi_f &= i_Y \circ \text{ev}_{X,Y} \circ (f \otimes 1_X) = \text{ev}_{13, Y^{\vee\vee}}^\phi \circ (\alpha_{Y^\vee, X^\vee}^{-1} \otimes 1_X) \circ (d_{X,Y} \otimes 1_X) \circ (f \otimes 1_X) \\
&= \text{ev}_{13, Y^{\vee\vee}}^\phi \circ (D_{\iota_f} \otimes 1_X).
\end{aligned}$$

Next, we have $i_{Y^\vee} \circ \varphi_g = \text{ev}_{13, Y^{\vee\vee\vee}}^\phi \circ (D_{\iota_g} \otimes 1_{X^\vee})$ by the previous computation (because $\alpha_{Y^{\vee\vee}, X^{\vee\vee}}$ is an isomorphism). Applying $(i_Y)^\vee$ we deduce, by Lemma 2.6,

$$\begin{aligned}
\varphi_g &= (i_Y)^\vee \circ \text{ev}_{13, Y^{\vee\vee\vee}}^\phi \circ (D_{\iota_g} \otimes 1_{X^\vee}) = \text{ev}_{13, Y^\vee} \circ (1_{X^{\vee\vee}} \otimes (i_Y)^\vee \otimes 1_{X^\vee}) \circ (D_{\iota_g} \otimes 1_{X^\vee}) \\
&= \text{ev}_{13, Y^\vee} \circ (i_X \otimes 1_{Y^\vee} \otimes 1_{X^\vee}) \circ (i_X^{-1} \otimes (i_Y)^\vee \otimes 1_{X^\vee}) \circ (D_{\iota_g} \otimes 1_{X^\vee}) \\
&= \text{ev}_{13, Y^\vee}^\tau \circ (i_X^{-1} \otimes (i_Y)^\vee \otimes 1_{X^\vee}) \circ (D_{\iota_g} \otimes 1_{X^\vee}).
\end{aligned}$$

But it follows from (25) that we have $D_{\iota_g^*} = (i_X^{-1} \otimes (i_Y)^\vee) \circ D_{\iota_g}$, proving that the second diagram is commutative. \square

2.2. Some commutative diagram involving the Casimir element. If we have given two objects X and Y and $W = W_1 \otimes W_2 \otimes W_3 \otimes W_4$, where (W_1, W_2, W_3, W_4) is a permutation of the string (X^\vee, X, Y^\vee, Y) , we define morphisms $\text{ev}_{ij,kl}^{\alpha,\beta} : W \rightarrow \mathbb{I}$, where $i, j, k, l \in \{1, 2, 3, 4\}$ are such that $i < j$, $k < l$ and $i < k$ and $\alpha, \beta \in \{\phi, \tau\}$, as follows. We let ij be one of the two pairs for which $\text{ev} : W_i \otimes W_j \rightarrow \mathbb{I}$ is defined and we write a corresponding superscript $\alpha = \phi$ if $W_j \in \{X, Y\}$ (so that $\text{ev} = \text{ev}_X$ or ev_Y) or $\alpha = \tau$ if $W_j \in \{X^\vee, Y^\vee\}$ (so that $\text{ev} = \text{ev}_X^\tau$ or ev_Y^τ); the same rule is applied to the triple (k, l, β) . Then we define

$$\text{ev}_{ij,kl}^{\alpha,\beta} : W \xrightarrow{\tau_\sigma} X^\vee \otimes X \otimes Y^\vee \otimes Y \xrightarrow{\text{ev}_X \otimes \text{ev}_Y} \mathbb{I} \otimes \mathbb{I} \xrightarrow{u_i} \mathbb{I},$$

where τ_σ is the morphism obtained from any permutation σ suitably reordering the factors. We have, for example,

$$\begin{aligned}
\text{ev}_{12,34}^{\phi,\phi} &: X^\vee \otimes X \otimes X^\vee \otimes X \xrightarrow{\text{ev}_X \otimes \text{ev}_X} \mathbb{I} \otimes \mathbb{I} \xrightarrow{u_1} \mathbb{I}. \\
\text{ev}_{14,23}^{\tau,\phi} &: X \otimes X^\vee \otimes X \otimes X^\vee \xrightarrow{\tau_{1,2} \otimes \tau_{3,4} \otimes 1_{X^\vee}} X^\vee \otimes X \otimes X \otimes X^\vee \xrightarrow{\text{ev}_X \otimes \text{ev}_X^\tau} \mathbb{I} \otimes \mathbb{I} \xrightarrow{u_1} \mathbb{I}.
\end{aligned}$$

We say that an object X admits a Casimir element if X is a reflexive object such that $\epsilon : X^{\vee\vee} \otimes X^\vee \rightarrow (X^\vee \otimes X)^\vee$ is an isomorphism. Then we define the Casimir element:

$$C_X : \mathbb{I} \xrightarrow{\text{ev}_X^\vee} (X^\vee \otimes X)^\vee \xrightarrow{\epsilon^{-1}} X^{\vee\vee} \otimes X^\vee \xrightarrow{i_X^{-1} \otimes 1_{X^\vee}} X \otimes X^\vee.$$

We collect in the following lemma well known properties of the Casimir element.

Lemma 2.8. *Suppose that X has a Casimir element.*

(1) C_X is the unique morphism making one of the following diagrams commutative:

$$\begin{aligned}
\text{ev}_X &: X^\vee \otimes X \xrightarrow{1_{X^\vee} \otimes C_X \otimes 1_X} X^\vee \otimes X \otimes X^\vee \otimes X \xrightarrow{\text{ev}_{12,34}^{\phi,\phi}} \mathbb{I}, \\
\text{ev}_X &: X^\vee \otimes X \xrightarrow{C_X \otimes \tau_{X^\vee, X}} X \otimes X^\vee \otimes X \otimes X^\vee \xrightarrow{\text{ev}_{14,23}^{\tau,\phi}} \mathbb{I}, \\
\text{ev}_X &: X^\vee \otimes X \xrightarrow{\tau_{X^\vee, X} \otimes C_X} X \otimes X^\vee \otimes X \otimes X^\vee \xrightarrow{\text{ev}_{14,23}^{\tau,\phi}} \mathbb{I}.
\end{aligned}$$

(2) C_X is the unique morphism making one of the following diagrams commutative:

$$\begin{aligned} 1_X : X &\xrightarrow{C_X \otimes 1_X} X \otimes X^\vee \otimes X \xrightarrow{1_X \otimes \text{ev}_X} X, \\ 1_{X^\vee} : X^\vee &\xrightarrow{1_{X^\vee} \otimes C_X} X^\vee \otimes X \otimes X^\vee \xrightarrow{\text{ev}_X \otimes 1_{X^\vee}} X^\vee. \end{aligned}$$

(3) If X_1, X_2 and $X_1 \otimes X_2$ have a Casimir element, then

$$C_{X_1 \otimes X_2} = (1_{X_1 \otimes X_2} \otimes \epsilon) \circ (1_{X_1} \otimes \tau_{X_1^\vee, X_2} \otimes 1_{X_2^\vee}) \circ (C_{X_1} \otimes C_{X_2}).$$

(4) If $X = X^+ \oplus X^-$ is a biproduct decomposition inducing $X^\vee = X^{\vee+} \oplus X^{\vee-}$, then X^\pm both have a Casimir element and $C_{X^\pm} = (p_{X^\pm} \otimes p_{X^{\vee\pm}}) \circ C_X$ for the associated surjective morphisms $p_{X^\pm} : X \rightarrow X^\pm$ and $p_{X^{\vee\pm}} : X^\vee \rightarrow X^{\vee\pm}$.

(5) We have that X^\vee has a Casimir element and $C_{X^\vee} = (1_{X^\vee} \otimes i_X) \circ \tau_{X, X^\vee} \circ C_X$.

When X has a Casimir element, an explicit inverse of the canonical map $f \mapsto \varphi_f$ can be given. This is the content of the subsequent proposition.

Proposition 2.9. Suppose that we have given $f : S \rightarrow \text{hom}(X, Y)$, which is associated to $\varphi_f : S \otimes X \rightarrow Y$, and that X has a Casimir element. Then the following diagram is commutative

$$\begin{array}{ccc} S & \xrightarrow{f} & \text{hom}(X, Y) \\ 1_S \otimes C_X \downarrow & & \uparrow \alpha_{X, Y} \\ S \otimes X \otimes X^\vee & \xrightarrow{\varphi_f \otimes 1_{X^\vee}} & Y \otimes X^\vee. \end{array}$$

Proof. Consider the following diagram

$$\begin{array}{ccccc} S \otimes X & \xrightarrow{1_S \otimes C_X \otimes 1_X} & S \otimes X \otimes X^\vee & \xrightarrow{\varphi_f \otimes 1_{X^\vee}} & Y \otimes X^\vee & \xrightarrow{\alpha_{X, Y} \otimes 1_X} & \text{hom}(X, Y) \otimes X \\ & \searrow 1_{S \otimes X} & \downarrow 1_{S \otimes X} \otimes \text{ev}_X & & \downarrow 1_Y \otimes \text{ev}_X & & \swarrow \text{ev}_{X, Y} \\ & & S \otimes X & \xrightarrow{\varphi_f} & Y & & \end{array}$$

The first triangle is commutative because $1_X : X \xrightarrow{C_X \otimes 1_X} X \otimes X^\vee \otimes X \xrightarrow{1_X \otimes \text{ev}_X} X$ by Lemma 2.8, the square is commutative by functoriality of \otimes and the second triangle by definition of $\alpha_{X, Y}$. But the map φ_a associated to $a := \alpha_{X, Y} \circ (\varphi_f \otimes 1_{X^\vee}) \circ (1_S \otimes C_X)$ is obtained going from $S \otimes X$ to $\text{hom}(X, Y) \otimes X$ in the upper row and then applying $\text{ev}_{X, Y}$. The commutativity implies that this is the morphism $\varphi_f \circ 1_{S \otimes X} = \varphi_f$. \square

We are mainly concerned with the following consequence of Proposition 2.9: when X has a Casimir element,

$$D_f : S \xrightarrow{1_S \otimes C_X} S \otimes X \otimes X^\vee \xrightarrow{\varphi_f \otimes 1_{X^\vee}} Y \otimes X^\vee. \quad (27)$$

2.3. Behavior of the internal multiplications with respect to tensor product constructions. We suppose in this section that we have given $f_i : S_i \rightarrow \text{hom}(X_i, Y_i)$ is associated to $\varphi_i = \varphi_{f_i} : S_i \otimes X_i \rightarrow Y_i$ for $i = 1, 2$. Define the following morphisms

$$\begin{aligned} f_1 \otimes_\epsilon f_2 &: S_1 \otimes S_2 \xrightarrow{f_1 \otimes f_2} \text{hom}(X_1, Y_1) \otimes \text{hom}(X_2, Y_2) \xrightarrow{\epsilon} \text{hom}(X_1 \otimes X_2, Y_1 \otimes Y_2), \\ f_1 \otimes_\epsilon^\tau f_2 &: S_2 \otimes S_1 \xrightarrow{\tau_{S_2, S_1}} S_1 \otimes S_2 \xrightarrow{f_1 \otimes f_2} \text{hom}(X_1, Y_1) \otimes \text{hom}(X_2, Y_2) \xrightarrow{\epsilon} \text{hom}(X_1 \otimes X_2, Y_1 \otimes Y_2), \\ \varphi_1 \otimes_\epsilon \varphi_2 &: S_1 \otimes S_2 \otimes X_1 \otimes X_2 \xrightarrow{1_{S_1} \otimes \tau_{2,3} \otimes 1_{X_2}} S_1 \otimes X_1 \otimes S_2 \otimes X_2 \xrightarrow{\varphi_1 \otimes \varphi_2} Y_1 \otimes Y_2, \\ \varphi_1 \otimes_\epsilon^\tau \varphi_2 &: S_2 \otimes S_1 \otimes X_1 \otimes X_2 \xrightarrow{\tau_{1,2} \otimes 3 \otimes 1_{X_2}} S_1 \otimes X_1 \otimes S_2 \otimes X_2 \xrightarrow{\varphi_1 \otimes \varphi_2} Y_1 \otimes Y_2. \end{aligned}$$

It is easy to see that one has the following result.

Lemma 2.10. We have that $\varphi_1 \otimes_\epsilon \varphi_2 = \varphi_{f_1 \otimes_\epsilon f_2}$ (resp. $\varphi_1 \otimes_\epsilon^\tau \varphi_2 = \varphi_{f_1 \otimes_\epsilon^\tau f_2}$) is the morphism associated to $f_1 \otimes_\epsilon f_2$ (resp. $f_1 \otimes_\epsilon^\tau f_2$).

Next, we consider the associated internal multiplication morphisms $\iota_i := \iota_{f_i}$, for $i = 1, 2$. The following lemma is easily deduced from the characterizing property (15) of φ_{ι_i} .

Lemma 2.11. *The following diagram is commutative*

$$\begin{array}{ccc}
S_1 \otimes S_2 \otimes X_1 \otimes X_2 \otimes Y_1^\vee \otimes Y_2^\vee & \xrightarrow{(1_{X_1} \otimes X_2 \otimes (\varphi_{\iota_1} \otimes_\epsilon \varphi_{\iota_2})) \circ (\tau_{S_1 \otimes S_2, X_1 \otimes X_2} \otimes 1_{Y_1^\vee \otimes Y_2^\vee})} & X_1 \otimes X_2 \otimes X_1^\vee \otimes X_2^\vee \\
(\varphi_1 \otimes_\epsilon \varphi_2) \otimes 1_{Y_1^\vee \otimes Y_2^\vee} \downarrow & & \downarrow \text{ev}_{13,24}^{\tau, \tau} \\
Y_1 \otimes Y_2 \otimes Y_1^\vee \otimes Y_2^\vee & \xrightarrow{\text{ev}_{13,24}^{\tau, \tau}} & \mathbb{I}
\end{array}$$

and the same with \otimes_ϵ replaced by \otimes_ϵ^τ and $S_1 \otimes S_2$ by $S_2 \otimes S_1$.

Remark 2.12. *It is easy to deduce from Lemma 2.11 that $\iota_{f_1 \otimes_\epsilon f_2} : S_1 \otimes S_2 \rightarrow \text{hom}((Y_1 \otimes Y_2)^\vee, (X_1 \otimes X_2)^\vee)$ is associated to $\varphi_{\iota_{f_1 \otimes_\epsilon f_2}}$ making the following diagram commutative:*

$$\begin{array}{ccc}
S_1 \otimes S_2 \otimes Y_1^\vee \otimes Y_2^\vee & \xrightarrow{\varphi_{\iota_{f_1}} \otimes_\epsilon \varphi_{\iota_{f_2}}} & X_1^\vee \otimes X_2^\vee \\
1_{S_1 \otimes S_2} \otimes \epsilon \downarrow & & \downarrow \epsilon \\
S_1 \otimes S_2 \otimes (Y_1 \otimes Y_2)^\vee & \xrightarrow{\varphi_{\iota_{f_1 \otimes_\epsilon f_2}}} & (X_1 \otimes X_2)^\vee.
\end{array}$$

In particular, when the ϵ morphisms are isomorphism, we deduce from Remark 2.2 that the commutativity of the diagram of Lemma 2.11 is characterizing for $\varphi_{\iota_1} \otimes_\epsilon \varphi_{\iota_2}$ (and similarly for $\varphi_{\iota_1} \otimes_\epsilon^\tau \varphi_{\iota_2}$)

From now on we specialize ourselves to the case where $f_1 : S_1 \rightarrow \text{hom}(X, Y)$ and $f_2 : S_2 \rightarrow \text{hom}(X^\vee, Y^\vee)$. We will assume, from now on, that $\alpha_{X, Y}$ and α_{X^\vee, Y^\vee} are isomorphisms, so that D_{f_1} and D_{f_2} are defined, and that X , X^\vee and Y have a Casimir element.

Lemma 2.13. *The following diagram is commutative:*

$$\begin{array}{ccccc}
S_1 \otimes S_2 & \xrightarrow{C_Y \otimes 1_{S_1 \otimes S_2} \otimes C_X} & Y \otimes Y^\vee \otimes S_1 \otimes S_2 \otimes X \otimes X^\vee & & \\
D_{f_1} \otimes D_{f_2} \downarrow & & \searrow 1_{Y^\vee \otimes Y} \otimes (\varphi_{f_1} \otimes_\epsilon \varphi_{f_2}) & & \\
Y \otimes X^\vee \otimes Y^\vee \otimes X^{\vee\vee} & \xrightarrow{\text{ev}_{24, Y \otimes Y^\vee}^\tau} & Y \otimes Y^\vee & \xrightarrow{C_Y \otimes 1_{Y \otimes Y^\vee}} & Y \otimes Y^\vee \otimes Y \otimes Y^\vee,
\end{array}$$

where

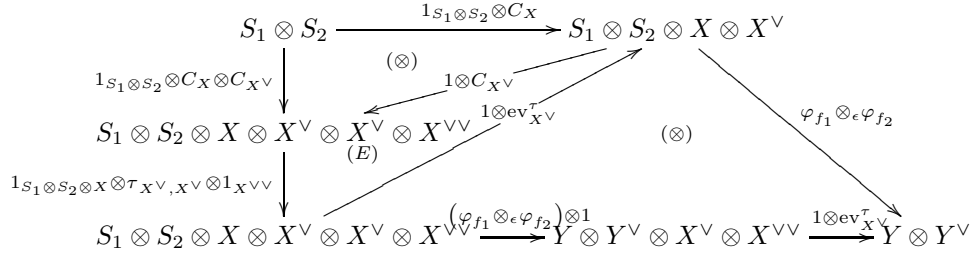
$$\text{ev}_{24, Y \otimes Y^\vee}^\tau : Y \otimes X^\vee \otimes Y^\vee \otimes X^{\vee\vee} \xrightarrow{1_Y \otimes \tau_{X^\vee, Y^\vee} \otimes 1_{X^{\vee\vee}}} Y \otimes Y^\vee \otimes X^\vee \otimes X^{\vee\vee} \xrightarrow{1_{Y \otimes Y^\vee} \otimes \text{ev}_{X^\vee}^\tau} Y \otimes Y^\vee.$$

Proof. Define

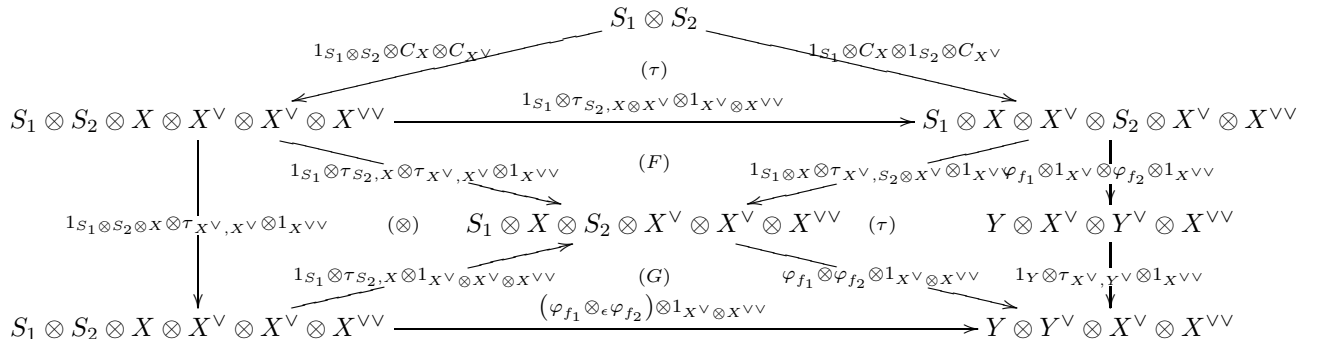
$$\begin{aligned}
D_{12} : S_1 \otimes S_2 & \xrightarrow{1_{S_1 \otimes S_2} \otimes C_X \otimes C_{X^\vee}} S_1 \otimes S_2 \otimes X \otimes X^\vee \otimes X^\vee \otimes X^{\vee\vee} \xrightarrow{1_{S_1 \otimes S_2} \otimes X \otimes \tau_{X^\vee, X^\vee} \otimes 1_{X^{\vee\vee}}} \\
& S_1 \otimes S_2 \otimes X \otimes X^\vee \otimes X^\vee \otimes X^{\vee\vee} \xrightarrow{(\varphi_{f_1} \otimes_\epsilon \varphi_{f_2}) \otimes 1_{X^\vee \otimes X^{\vee\vee}}} Y \otimes Y^\vee \otimes X^\vee \otimes X^{\vee\vee}
\end{aligned}$$

$$\begin{array}{c}
Y \otimes Y^\vee \otimes S_1 \otimes S_2 \otimes X \otimes X^\vee \xrightarrow{1_{Y \otimes Y^\vee} \otimes (\varphi_{f_1} \otimes \epsilon \varphi_{f_2})} Y \otimes Y^\vee \otimes Y \otimes Y^\vee \\
C_Y \otimes 1_{S_1 \otimes S_2} \otimes C_X \uparrow \quad \quad \quad (A) \quad \quad \quad \uparrow C_Y \otimes 1_{Y \otimes Y^\vee} \\
S_1 \otimes S_2 \xrightarrow{1_{S_1 \otimes S_2} \otimes C_X} S_1 \otimes S_2 \otimes X \otimes X^\vee \xrightarrow{\varphi_{f_1} \otimes \epsilon \varphi_{f_2}} Y \otimes Y^\vee \\
\searrow D_{12} \quad \quad \quad (B) \quad \quad \quad \nearrow 1_{Y \otimes Y^\vee} \otimes \text{ev}_{X^\vee}^\tau \\
(C) \quad Y \otimes Y^\vee \otimes X^\vee \otimes X^{\vee\vee} \quad (D) \\
\downarrow D_{f_1} \otimes D_{f_2} \quad \quad \quad \uparrow 1_{Y \otimes \tau_{X^\vee, Y^\vee}} \otimes 1_{X^{\vee\vee}} \quad \quad \quad \nwarrow \text{ev}_{24, Y \otimes Y^\vee}^\tau \\
Y \otimes X^\vee \otimes Y^\vee \otimes X^{\vee\vee}
\end{array}$$
$$(C_Y \otimes 1_{Y \otimes Y^\vee}) \circ (\varphi_{f_1} \otimes_\epsilon \varphi_{f_2}) \circ (1_{S_1 \otimes S_2} \otimes C_X) = (1_{Y \otimes Y^\vee} \otimes (\varphi_{f_1} \otimes_\epsilon \varphi_{f_2})) \circ (C_Y \otimes 1_{S_1 \otimes S_2} \otimes C_X).$$

Region (B) is commutative. Consider the following diagram


$$\begin{array}{ccc}
X^{\vee} \xrightarrow{C_{X^{\vee} \otimes 1_{X^{\vee}}}} X^{\vee} \otimes X^{\vee \vee} \otimes X^{\vee} & \xrightarrow{1_{X^{\vee}} \otimes \text{ev}_{X^{\vee}}} & X^{\vee} \\
\downarrow 1_{X^{\vee}} \otimes C_{X^{\vee}} \tau_{X^{\vee} \otimes X^{\vee \vee}, X^{\vee}} & \searrow 1_{X^{\vee}} \otimes \tau_{X^{\vee \vee}, X^{\vee}} & \nearrow 1_{X^{\vee}} \otimes \text{ev}_{X^{\vee}}^{\tau} \\
X^{\vee} \otimes X^{\vee} \otimes X^{\vee} & \xrightarrow{\tau_{X^{\vee}, X^{\vee}} \otimes 1_{X^{\vee}}} & X^{\vee} \otimes X^{\vee} \otimes X^{\vee \vee}
\end{array}$$

Region (C) is commutative. Consider the following diagram



Going around this diagram clockwise (resp. counter-clockwise) from $S_1 \otimes S_2$ until $Y \otimes Y^\vee \otimes X^\vee \otimes X^{\vee\vee}$ we find $(1_Y \otimes \tau_{X^\vee, Y^\vee} \otimes 1_{X^{\vee\vee}}) \circ (D_{f_1} \otimes D_{f_2})$ (resp. D_{12}) because, by (27), we have $D_{f_i} = (\varphi_{f_i} \otimes 1_{X_i^\vee}) \circ (1_{S_i} \otimes C_{X_i})$ for $i = 1, 2$, where $X_1 = X$ and $X_2 = X^\vee$ (resp. by definition of D_{12}). It follows that we have to show that the external portion of this diagram is commutative. The region (F) is commutative by an explicit computation of the involved permutations, while (G) by definition of $(\varphi_{f_1} \otimes_\epsilon \varphi_{f_2})$. \square

In addition to the other assumptions we will assume in the following proposition that α_{Y^\vee, X^\vee} and $\alpha_{Y^{\vee\vee}, X^{\vee\vee}}$ are isomorphisms, so that $D_{\iota_{f_1}}$ and $D_{\iota_{f_2}}$ are defined, and that $Y^\vee, Y^{\vee\vee}$ have a Casimir element.

Proposition 2.14. *The following diagrams are commutative.*

(1)

$$\begin{array}{ccc} S_1 \otimes S_2 & \xrightarrow{C_Y \otimes 1_{S_1 \otimes S_2} \otimes C_X} & Y \otimes Y^\vee \otimes S_1 \otimes S_2 \otimes X \otimes X^\vee \\ \downarrow D_{f_1} \otimes D_{f_2} & & \downarrow 1_{Y^\vee \otimes Y} \otimes (\varphi_{f_1} \otimes_\epsilon \varphi_{f_2}) \\ Y \otimes X^\vee \otimes Y^\vee \otimes X^{\vee\vee} & \xrightarrow{\text{ev}_{13,24}^{\tau, \tau}} \mathbb{I} \xleftarrow{\text{ev}_{14,23}^{\tau, \phi}} & Y \otimes Y^\vee \otimes Y \otimes Y^\vee \end{array}$$

(2)

$$\begin{array}{ccc} S_1 \otimes S_2 & \xrightarrow{C_{X^\vee} \otimes 1_{S_1 \otimes S_2} \otimes C_{Y^\vee}} & X^\vee \otimes X^{\vee\vee} \otimes S_1 \otimes S_2 \otimes Y^\vee \otimes Y^{\vee\vee} \\ \downarrow D_{\iota_{f_1}} \otimes D_{\iota_{f_2}} & & \downarrow 1_{X^\vee \otimes X^{\vee\vee}} \otimes (\varphi_{\iota_{f_1}} \otimes_\epsilon \varphi_{\iota_{f_2}}) \\ X^\vee \otimes Y^{\vee\vee} \otimes X^{\vee\vee} \otimes Y^{\vee\vee\vee} & \xrightarrow{\text{ev}_{13,24}^{\tau, \tau}} \mathbb{I} \xleftarrow{\text{ev}_{14,23}^{\tau, \phi}} & X^\vee \otimes X^{\vee\vee} \otimes X^\vee \otimes X^{\vee\vee} \end{array}$$

(3)

$$\begin{array}{ccc} S_1 \otimes S_2 & \xrightarrow{1_{S_1 \otimes S_2} \otimes C_X \otimes C_{Y^\vee}} & S_1 \otimes S_2 \otimes X \otimes X^\vee \otimes Y^\vee \otimes Y^{\vee\vee} \\ \downarrow D_{\iota_{f_1}} \otimes D_{\iota_{f_2}} & & \downarrow (\varphi_{f_1} \otimes_\epsilon \varphi_{f_2}) \otimes 1_{Y^\vee \otimes Y^{\vee\vee}} \\ X^\vee \otimes Y^{\vee\vee} \otimes X^{\vee\vee} \otimes Y^{\vee\vee\vee} & \xrightarrow{\text{ev}_{13,24}^{\tau, \tau}} \mathbb{I} \xleftarrow{\text{ev}_{13,24}^{\tau, \tau}} & Y \otimes Y^\vee \otimes Y^\vee \otimes Y^{\vee\vee} \end{array}$$

Proof. (1) According to Lemma 2.13 we have

$$\text{ev}_{14,23}^{\tau, \phi} \circ (1_{Y^\vee \otimes Y} \otimes (\varphi_{f_1} \otimes_\epsilon \varphi_{f_2})) \circ (C_Y \otimes 1_{S_1 \otimes S_2} \otimes C_X) = \text{ev}_{14,23}^{\tau, \phi} \circ (C_Y \otimes 1_{Y \otimes Y^\vee}) \circ \text{ev}_{24, Y \otimes Y^\vee}^\tau \circ (D_{f_1} \otimes D_{f_2}).$$

It follows from Lemma 2.8 (1) we have $\text{ev}_Y = \text{ev}_{14,23}^{\tau, \phi} \circ (C_Y \otimes \tau_{Y^\vee, Y}) = \text{ev}_{14,23}^{\tau, \phi} \circ (C_Y \otimes 1_{Y \otimes Y^\vee}) \circ \tau_{Y^\vee, Y}$ and, by definition, $\text{ev}_{24, Y \otimes Y^\vee}^\tau = (1_{Y \otimes Y^\vee} \otimes \text{ev}_{X^\vee}^\tau) \circ (1_Y \otimes \tau_{X^\vee, Y^\vee} \otimes 1_{X^{\vee\vee}})$. We deduce

$$\begin{aligned} \text{ev}_{14,23}^{\tau, \phi} \circ (C_Y \otimes 1_{Y \otimes Y^\vee}) \circ \text{ev}_{24, Y \otimes Y^\vee}^\tau &= \text{ev}_{14,23}^{\tau, \phi} \circ (C_Y \otimes 1_{Y \otimes Y^\vee}) \circ \tau_{Y^\vee, Y} \circ \tau_{Y, Y^\vee} \circ \text{ev}_{24, Y \otimes Y^\vee}^\tau \\ &= \text{ev}_Y \circ \tau_{Y, Y^\vee} \circ (1_{Y \otimes Y^\vee} \otimes \text{ev}_{X^\vee}^\tau) \circ (1_Y \otimes \tau_{X^\vee, Y^\vee} \otimes 1_{X^{\vee\vee}}) = \text{ev}_{13,24}^{\tau, \tau}. \end{aligned}$$

Hence we find

$$\text{ev}_{14,23}^{\tau, \phi} \circ (1_{Y^\vee \otimes Y} \otimes (\varphi_{f_1} \otimes_\epsilon \varphi_{f_2})) \circ (C_Y \otimes 1_{S_1 \otimes S_2} \otimes C_X) = \text{ev}_{13,24}^{\tau, \tau} \circ (D_{f_1} \otimes D_{f_2}).$$

(2) This is just our claim 1. applied to the couple $(\iota_{f_1}, \iota_{f_2})$ rather than (f_1, f_2) .

[illegible]

Commutativity of the labeled regions of (28). The region (A) is commutative by our claim (2), the region (C) by the defining property of i_X , the region (D) by the definitions of $\text{ev}_{14,23}^{\tau_1,\tau}$ and $\text{ev}_{13,24}^{\tau_1,\tau}$. The commutativity of the region (B) follows the equality $C_{X^\vee} = (1_{X^\vee} \otimes i_X) \circ \tau_{X,X^\vee} \circ C_X$ of Lemma 2.8 (5), from which we deduce that

3. A FORMAL POINCARÉ DUALITY ISOMORPHISM

$$End(\mathbb{I}) \rightarrow End(W)$$
$$\lambda_W : W \xrightarrow{l_W} \mathbb{I} \otimes W \xrightarrow{\lambda \otimes 1_W} \mathbb{I} \otimes W \xrightarrow{l_W^{-1}} W, \lambda \in \text{End}(\mathbb{I}).$$

Suppose in this section that \mathcal{C} is rigid. We assume that we have given morphisms $f_{S,X} : S \rightarrow \text{hom}(X, Y)$, $f_{X,S} : X \rightarrow \text{hom}(S, Y)$, $f_{S^\vee, X^\vee} : S^\vee \rightarrow \text{hom}(X^\vee, Y^\vee)$ and $f_{X^\vee, S^\vee} : X^\vee \rightarrow \text{hom}(S^\vee, Y^\vee)$. We write $\varphi_{S,X} : S \otimes X \rightarrow Y$, $\varphi_{X,S} : X \otimes S \rightarrow Y$, $\varphi_{S^\vee, X^\vee} : S^\vee \otimes X^\vee \rightarrow Y^\vee$ and $\varphi_{X^\vee, S^\vee} : X^\vee \otimes S^\vee \rightarrow Y^\vee$ for the associated morphisms. We set $\iota_{S,X} := \iota_{f_{S,X}}$, $\iota_{X,S} := \iota_{f_{X,S}}$, $\iota_{S^\vee, X^\vee} := \iota_{f_{S^\vee, X^\vee}}$ and $\iota_{X^\vee, S^\vee} := \iota_{f_{X^\vee, S^\vee}}$. We may consider:

$$D_{S,X^\vee} := D_{\iota_{S,X}} : S \rightarrow X^\vee \otimes Y^{\vee\vee}, D_{X,S^\vee} := D_{\iota_{X,S}} : X \rightarrow S^\vee \otimes Y^{\vee\vee},$$

$$D_{\iota_{S^\vee,X^\vee}} : S^\vee \rightarrow X^{\vee\vee} \otimes Y^{\vee\vee\vee} \text{ and } D_{\iota_{X^\vee,S^\vee}} : X^\vee \rightarrow S^{\vee\vee} \otimes Y^{\vee\vee\vee},$$

$$D_{S^\vee, X} := D_{\iota_{S^\vee, X}^*} : S^\vee \rightarrow X \otimes Y^\vee \text{ and } D_{X^\vee, S} := D_{\iota_{X^\vee, S}^*} : X^\vee \rightarrow S \otimes Y^\vee.$$
$$(Cas)_{\mu_{S,X}} : \mu_{S,X} \cdot C_S = \left(\varphi_{i_{X^V, S^V}}^* \otimes_{\epsilon}^{\tau} \varphi_{i_{X,S}} \right) \circ (C_X \otimes C_Y) \Leftrightarrow \mu_{S,X} \cdot C_{S^V} = \left(\varphi_{i_{X,S}} \otimes_{\epsilon} \varphi_{i_{X^V, S^V}} \right) \circ (C_X \otimes C_{Y^V})$$

16

for some $\mu_{S,X} \in \text{End}(\mathbb{I})$. Exchanging the roles of S and X we also have, for some $\mu_{X,S} \in \text{End}(\mathbb{I})$,

$$(Cas)_{\mu_{X,S}} : \mu_{X,S} \cdot C_X = \left(\varphi_{\iota_{S^\vee, X^\vee}}^* \otimes_\epsilon^\tau \varphi_{\iota_{S,X}} \right) \circ (C_S \otimes C_Y) \Leftrightarrow \mu_{X,S} \cdot C_{X^\vee} = \left(\varphi_{\iota_{S,X}} \otimes_\epsilon \varphi_{\iota_{S^\vee, X^\vee}} \right) \circ (C_S \otimes C_{Y^\vee}).$$

If $(V, W) = (S, X)$ or (X, S) and $\lambda_{V,W}, \lambda_{V^\vee, W^\vee} \in \text{End}(\mathbb{I})$, we will consider the following diagrams:

$$\begin{array}{ccc} V \otimes W & \xrightarrow{\tau_{V,W}} & W \otimes V \\ \varphi_{V,W} \downarrow & (Com)_{\lambda_{V,W}} & \downarrow \varphi_{W,V} \\ Y & \xrightarrow{\lambda_{V,W}} & Y \end{array} \quad \begin{array}{ccc} V^\vee \otimes W^\vee & \xrightarrow{\tau_{V^\vee, W^\vee}} & W^\vee \otimes V^\vee \\ \varphi_{V^\vee, W^\vee} \downarrow & (Com)_{\lambda_{V^\vee, W^\vee}} & \downarrow \varphi_{W^\vee, V^\vee} \\ Y^\vee & \xrightarrow{\lambda_{V^\vee, W^\vee}} & Y^\vee \end{array}$$

It will be convenient to introduce the following shorthand. We set $[W] := W \otimes W^\vee$, $\varphi_{[S],[X]} := \varphi_{S,X} \otimes_\epsilon \varphi_{S^\vee, X^\vee}$, $\varphi_{[X],[S]} := \varphi_{X,S} \otimes_\epsilon \varphi_{X^\vee, S^\vee}$, $\varphi_{\iota_{[S],[X]}} := \varphi_{\iota_{S,X}} \otimes_\epsilon \varphi_{\iota_{S^\vee, X^\vee}}$, $\varphi_{\iota_{[X],[S]}} := \varphi_{\iota_{X,S}} \otimes_\epsilon \varphi_{\iota_{X^\vee, S^\vee}}$, $\varphi_{\iota_{[S],[X]}}^\tau := \varphi_{\iota_{S^\vee, X^\vee}}^* \otimes_\epsilon^\tau \varphi_{\iota_{S,X}}$ and $\varphi_{\iota_{[X],[S]}}^\tau := \varphi_{\iota_{X^\vee, S^\vee}}^* \otimes_\epsilon^\tau \varphi_{\iota_{X,S}}$. If $(V, W) = (S, X)$ or (X, S) and $\lambda_{[V],[W]} \in \text{End}(\mathbb{I})$, we will consider the following diagram:

$$\begin{array}{ccc} [V] \otimes [W] & \xrightarrow{\tau_{[V],[W]}} & [W] \otimes [V] \\ \varphi_{[V],[W]} \downarrow & (Com)_{\lambda_{[V],[W]}} & \downarrow \varphi_{[W],[V]} \\ [Y] & \xrightarrow{\lambda_{[V],[W]}} & [Y] \end{array}$$

Remark 3.1. We have

$$(Com)_{\lambda_{V,W}} \text{ and } (Com)_{\lambda_{V^\vee, W^\vee}} \text{ commutative} \Rightarrow (Com)_{\lambda_{[V],[W]}} \text{ with } \lambda_{[V],[W]} = \lambda_{V,W} \cdot \lambda_{V^\vee, W^\vee}.$$

Proposition 3.2. If $(Cas)_{\mu_{S,X}}$ is satisfied and $(Com)_{\lambda_{[S],[X]}}$ is commutative, then the following diagrams are commutative:

$$\begin{array}{ccc} S \otimes S^\vee & \xrightarrow{\mu_{S,X} \cdot \text{ev}_S^\tau} & \mathbb{I} \\ D_{S,X^\vee} \otimes D_{S^\vee, X} \downarrow & & \\ X^\vee \otimes Y^{\vee\vee} \otimes X \otimes Y^\vee & \xrightarrow{\lambda_{[S],[X]} \cdot \text{ev}_{13,24}^{\phi, \phi}} & \mathbb{I} \end{array} \quad \begin{array}{ccc} S^\vee \otimes S & \xrightarrow{\mu_{S,X} \cdot \text{ev}_S} & \mathbb{I} \\ D_{S^\vee, X} \otimes D_{S, X^\vee} \downarrow & & \\ X \otimes Y^\vee \otimes X^\vee \otimes Y^{\vee\vee} & \xrightarrow{\lambda_{[S],[X]} \cdot \text{ev}_{13,24}^{\tau, \tau}} & \mathbb{I} \end{array}$$

Similarly if $(Cas)_{\mu_{X,S}}$ is satisfied and $(Com)_{\lambda_{[X],[S]}}$ is commutative, we get the analogue commutative diagram where (S, X) is replaced by (X, S) .

Proof. It is clear that the two diagrams are equivalently commutative, so that suffices to prove the commutativity of the first diagram. It follows from Proposition 2.14 (3) that we have

$$\lambda_{[S],[X]} \cdot \text{ev}_{13,24}^{\tau, \tau} \circ \left(D_{\iota_{S,X}} \otimes D_{\iota_{S^\vee, X^\vee}} \right) = \lambda_{[S],[X]} \cdot \text{ev}_{13,24}^{\tau, \tau} \circ \left(\varphi_{[S] \otimes [X]} \otimes 1_{[Y^\vee]} \right) \circ \left(1_{[S]} \otimes C_X \otimes C_{Y^\vee} \right).$$

In order to compute the right hand side, consider the following diagram:

$$\begin{array}{c} \begin{array}{c} [S] \otimes [X] \otimes [Y^\vee] \xrightarrow{\varphi_{[S] \otimes [X]} \otimes 1_{[Y^\vee]}} [X] \otimes [S] \otimes [Y^\vee] \xrightarrow{\varphi_{[X],[S]} \otimes 1_{[Y^\vee]}} [S] \otimes [X] \otimes [Y^\vee] \\ \downarrow \lambda_{[S],[X]} \quad \quad \quad \downarrow \lambda_{[X],[S]} \quad \quad \quad \downarrow \lambda_{[S],[X]} \\ [Y] \otimes [Y^\vee] \xrightarrow{\lambda_{[S],[X]}} [Y] \otimes [Y^\vee] \end{array} \\ \downarrow \text{ev}_{13,24}^{\tau, \tau} \\ \mathbb{I} \end{array} \quad \begin{array}{c} [S] \xrightarrow{i_S \otimes 1_{S^\vee}} S^{\vee\vee} \otimes S^\vee \xrightarrow{\tau_{S^{\vee\vee}, S^\vee}} S^\vee \otimes S^{\vee\vee} \\ \downarrow \mu_{S,X} \cdot 1_{S^{\vee\vee} \otimes S^\vee} \otimes C_{S^\vee} \quad \quad \downarrow \mu_{S,X} \cdot 1_{S^\vee \otimes S^{\vee\vee}} \otimes C_{S^\vee} \\ [S] \otimes [S^\vee] \xrightarrow{i_S \otimes 1_{S^\vee} \otimes 1_{[S^\vee]}} S^{\vee\vee} \otimes S^\vee \otimes [S^\vee] \xrightarrow{\tau_{S^{\vee\vee}, S^\vee} \otimes 1_{[S^\vee]}} S^\vee \otimes S^{\vee\vee} \otimes [S^\vee] \\ \downarrow \text{ev}_{13,24}^{\tau, \tau} \quad \quad \quad \downarrow \text{ev}_{14,23}^{\tau, \phi} \\ \mathbb{I} \end{array}$$

17

Here (A) is commutative by the adjoint property of Lemma 2.11, (B) = $(Com)_{\lambda_{[S],[X]} \otimes 1_{[Y^\vee]}}$ is commutative by assumption, the equality $\mu_{S,X} \cdot 1_{[S]} \otimes C_{S^\vee} = (1_{[S]} \otimes \varphi_{\iota_{[X],[S]}}) \circ (1_{[S]} \otimes C_X \otimes C_{Y^\vee})$ is assured by $(Cas)_{\mu_{S,X}}$, (C) is commutative by definition of i_S , (D) is clearly commutative and (E) by definition of $\text{ev}_{13,24}^{\phi,\tau}$ and $\text{ev}_{14,23}^{\tau,\phi}$. We deduce the first of the subsequent equalities, while that second follows from $\text{ev}_{14,23}^{\tau,\phi} \circ (\tau_{S^\vee, S^\vee} \otimes C_{S^\vee}) = \text{ev}_{S^\vee}$ granted by Lemma 2.8 (1) and the third by definition of i_S :

$$\begin{aligned} \lambda_{[S],[X]} \cdot \text{ev}_{13,24}^{\tau,\tau} \circ (D_{\iota_{S,X}} \otimes D_{\iota_{S^\vee,X^\vee}}) &= \mu_{S,X} \cdot \text{ev}_{14,23}^{\tau,\phi} \circ (\tau_{S^\vee, S^\vee} \otimes C_{S^\vee}) \circ (i_S \otimes 1_{S^\vee}) \\ &= \mu_{S,X} \cdot \text{ev}_{S^\vee} \circ (i_S \otimes 1_{S^\vee}) = \mu_{S,X} \cdot \text{ev}_S^\tau. \end{aligned}$$

We end the proof of the proposition by rearranging the left hand side of this equality by looking at the following diagram:

$$\begin{array}{ccc} & X^\vee \otimes Y^{\vee\vee} \otimes X^{\vee\vee} \otimes Y^{\vee\vee\vee} & \\ \nearrow D_{\iota_{S,X}} \otimes D_{\iota_{S^\vee,X^\vee}} & \uparrow (F) \quad 1_{X^\vee \otimes Y^{\vee\vee}} \otimes i_X \otimes i_{Y^\vee}(G) & \searrow \text{ev}_{13,24}^{\tau,\tau} \\ S \otimes S^\vee & \xrightarrow{D_{\iota_{S,X}} \otimes D_{\iota_{S^\vee,X^\vee}}} X^\vee \otimes Y^{\vee\vee} \otimes X \otimes Y^\vee & \xrightarrow{\text{ev}_{13,24}^{\phi,\phi}} \mathbb{I}. \end{array}$$

Here (F) is commutative by (26), while (G) is commutative by definition of i_W for $W = X$ and Y . The claimed commutativity follows.

Since the roles of S and X are symmetric, we get the same commutative diagram where (S, X) is replaced by (X, S) if $(Cas)_{\mu_{X,S}}$ is satisfied and $(Com)_{\lambda_{[X],[S]}}$ is commutative. \square

Lemma 3.3. *If $(Com)_{\lambda_{X^\vee, S^\vee}}$ is commutative, the following diagrams are commutative:*

$$\begin{array}{ccc} X^\vee \otimes S^\vee & \xrightarrow{D_{X^\vee, S^\vee} \otimes 1_{S^\vee}} & S \otimes Y^\vee \otimes S^\vee \\ \downarrow 1_{X^\vee} \otimes D_{S^\vee, X} & & \downarrow \lambda_{X^\vee, S^\vee} \cdot \text{ev}_{13, Y^\vee}^\tau \\ X^\vee \otimes X \otimes Y^\vee & \xrightarrow{\text{ev}_X \otimes 1_{Y^\vee}} & Y^\vee, \end{array} \quad \begin{array}{ccc} S^\vee \otimes X^\vee & \xrightarrow{1_{S^\vee} \otimes D_{X^\vee, S^\vee}} & S^\vee \otimes S \otimes Y^\vee \\ \downarrow D_{S^\vee, X^\vee} \otimes 1_{X^\vee} & & \downarrow \lambda_{X^\vee, S^\vee} \cdot \text{ev}_S \otimes 1_{Y^\vee} \\ X \otimes Y^\vee \otimes X^\vee & \xrightarrow{\text{ev}_{13, Y^\vee}^\tau} & Y^\vee. \end{array}$$

Similarly, if $(Com)_{\lambda_{S^\vee, X^\vee}}$ is commutative, we get the analogue commutative diagram where (S, X) is replaced by (X, S) .

If $(Com)_{\lambda_{X, S}}$ is commutative, the following diagrams are commutative:

$$\begin{array}{ccc} X \otimes S & \xrightarrow{D_{X, S^\vee} \otimes 1_S} & S^\vee \otimes Y^{\vee\vee} \otimes S \\ \downarrow 1_X \otimes D_{S, X^\vee} & & \downarrow \lambda_{X, S} \cdot \text{ev}_{13, Y^{\vee\vee}}^\phi \\ X \otimes X^\vee \otimes Y^{\vee\vee} & \xrightarrow{\text{ev}_X^\tau \otimes 1_{Y^{\vee\vee}}} & Y^{\vee\vee}, \end{array} \quad \begin{array}{ccc} S \otimes X & \xrightarrow{1_S \otimes D_{X, S^\vee}} & S \otimes S^\vee \otimes Y^{\vee\vee} \\ \downarrow D_{S, X^\vee} \otimes 1_X & & \downarrow \lambda_{X, S} \cdot \text{ev}_S^\tau \otimes 1_{Y^{\vee\vee}} \\ X^\vee \otimes Y^{\vee\vee} \otimes X & \xrightarrow{\text{ev}_{13, Y^{\vee\vee}}^\phi} & Y^{\vee\vee}. \end{array}$$

Similarly, if $(Com)_{\lambda_{S, X}}$ is commutative, we get the analogue commutative diagram where (S, X) is replaced by (X, S) .

Proof. It is clear that the first two diagrams are equivalently commutative, so that suffices to prove the commutativity of the first diagram. Consider the following diagram:

$$\begin{array}{ccccccc}
S^\vee \otimes X^\vee & \xleftarrow{\tau_{X^\vee, S^\vee}} & X^\vee \otimes S^\vee & \xrightarrow{\tau_{X^\vee, S^\vee}} & S^\vee \otimes X^\vee & \xrightarrow{\tau_{S^\vee, X^\vee}} & X^\vee \otimes S^\vee \\
\downarrow 1_{S^\vee} \otimes \varphi_{X^\vee, S^\vee} \otimes C_Y & (\tau) & \downarrow 1_{X^\vee} \otimes S^\vee \otimes C_Y & (\tau) & \downarrow 1_{S^\vee} \otimes X^\vee \otimes C_Y & (\tau) & \downarrow 1_{X^\vee} \otimes S^\vee \otimes C_Y \\
S^\vee \otimes X^\vee \otimes Y \otimes Y^\vee & \xleftarrow{\tau_{X^\vee, S^\vee} \otimes 1_{Y \otimes Y^\vee}} & X^\vee \otimes S^\vee \otimes Y \otimes Y^\vee & \xrightarrow{\tau_{X^\vee, S^\vee} \otimes 1_{Y \otimes Y^\vee}} & S^\vee \otimes X^\vee \otimes Y \otimes Y^\vee & \xrightarrow{\tau_{S^\vee, X^\vee} \otimes 1_{Y \otimes Y^\vee}} & X^\vee \otimes S^\vee \otimes Y \otimes Y^\vee \\
\downarrow 1_{S^\vee} \otimes \varphi_{X^\vee, S^\vee} \otimes 1_{Y^\vee} & & \downarrow \varphi_{X^\vee, S^\vee} \otimes 1_{Y \otimes Y^\vee} & (B) & \downarrow \varphi_{S^\vee, X^\vee} \otimes 1_{Y \otimes Y^\vee} & & \downarrow 1_{X^\vee} \otimes \varphi_{S^\vee, X^\vee} \otimes 1_{Y^\vee} \\
S^\vee \otimes S \otimes Y^\vee & (A) & Y^\vee \otimes Y \otimes Y^\vee & \xrightarrow{\lambda_{X^\vee, S^\vee}} & Y^\vee \otimes Y \otimes Y^\vee & (A) & X^\vee \otimes X \otimes Y^\vee \\
\downarrow \text{ev}_S \otimes 1_{Y^\vee} & & \downarrow \text{ev}_Y \otimes 1_{Y^\vee} & & \downarrow \text{ev}_Y \otimes 1_{Y^\vee} & & \downarrow \text{ev}_X \otimes 1_{Y^\vee} \\
Y^\vee & \xleftarrow{\text{ev}_Y \otimes 1_{Y^\vee}} & Y^\vee & \xrightarrow{\lambda_{X^\vee, S^\vee}} & Y^\vee & \xrightarrow{\text{ev}_Y \otimes 1_{Y^\vee}} & Y^\vee
\end{array}$$

The regions (A) are commutative by the adjoint property (23) of φ_{X^\vee, S^\vee} and φ_{S^\vee, X^\vee} , while (B) = $(Com)_{\lambda_{X^\vee, S^\vee} \otimes 1_{Y \otimes Y^\vee}}$ by our assumption. Recalling that $D_{X^\vee, S^\vee}^* = (\varphi_{X^\vee, S^\vee}^* \otimes 1_{Y^\vee}) \circ (1_{X^\vee} \otimes C_Y)$ and $D_{S^\vee, X^\vee}^* = (\varphi_{S^\vee, X^\vee}^* \otimes 1_{Y^\vee}) \circ (1_{S^\vee} \otimes C_Y)$ by (27), we deduce

$$\begin{aligned}
(\text{ev}_X \otimes 1_{Y^\vee}) \circ (1_{X^\vee} \otimes D_{S^\vee, X^\vee}^*) &= (\text{ev}_X \otimes 1_{Y^\vee}) \circ (1_{X^\vee} \otimes \varphi_{S^\vee, X^\vee}^* \otimes 1_{Y^\vee}) \circ (1_{X^\vee} \otimes S^\vee \otimes C_Y) \circ \tau_{S^\vee, X^\vee} \circ \tau_{X^\vee, S^\vee} \\
&= \lambda_{X^\vee, S^\vee} \cdot (\text{ev}_S \otimes 1_{Y^\vee}) \circ (1_{S^\vee} \otimes \varphi_{X^\vee, S^\vee}^* \otimes 1_{Y^\vee}) \circ (1_{S^\vee} \otimes X^\vee \otimes C_Y) \circ \tau_{X^\vee, S^\vee} \\
&= \lambda_{X^\vee, S^\vee} \cdot (\text{ev}_S \otimes 1_{Y^\vee}) \circ (1_{S^\vee} \otimes D_{X^\vee, S^\vee}^*) \circ \tau_{X^\vee, S^\vee} \\
&= \lambda_{X^\vee, S^\vee} \circ (\text{ev}_S \otimes 1_{Y^\vee}) \circ \tau_{S \otimes Y^\vee, S^\vee} \circ (D_{X^\vee, S^\vee}^* \otimes 1_{S^\vee}) \\
&= \lambda_{X^\vee, S^\vee} \circ \text{ev}_{13, Y^\vee}^\tau \circ (D_{X^\vee, S^\vee}^* \otimes 1_{S^\vee}).
\end{aligned}$$

The commutativity of the other diagrams is proved in a similar way. \square

Remark 3.4. If Y is a reflexive object the canonical morphism

$$\text{Hom}(X, Y) \rightarrow \text{Hom}(Y^\vee \otimes X, \mathbb{I})$$

mapping $f : X \rightarrow Y$ to $\text{ev}_{Y, \mathbb{I}} \circ (1_{Y^\vee} \otimes f) : Y^\vee \otimes X \xrightarrow{1_{Y^\vee} \otimes f} Y^\vee \otimes Y \xrightarrow{\text{ev}_{Y, \mathbb{I}}} \mathbb{I}$ is a bijection.

We can now prove the main result of this section.

Theorem 3.5. Suppose that $(Cas)_{\mu_{S, X}}$ is satisfied and that $(Com)_{\lambda_{[S], [X]}}$ is commutative. Then we have

$$\begin{aligned}
\mu_{S, X} : S &\xrightarrow{D_{S, X^\vee}} X^\vee \otimes Y^{\vee\vee} \xrightarrow{D_{X^\vee, S^\vee} \otimes 1_{Y^{\vee\vee}}} S \otimes Y^\vee \otimes Y^{\vee\vee} \xrightarrow{1_S \otimes \text{ev}_{Y^\vee}^\tau} S \xrightarrow{\lambda_{[S], [X]} \lambda_{X^\vee, S^\vee}} S, \\
&\text{if } (Com)_{\lambda_{X^\vee, S^\vee}} \text{ is commutative,} \\
\mu_{S, X} \lambda_{S^\vee, X^\vee} : S &\xrightarrow{D_{S, X^\vee}} X^\vee \otimes Y^{\vee\vee} \xrightarrow{D_{X^\vee, S^\vee} \otimes 1_{Y^{\vee\vee}}} S \otimes Y^\vee \otimes Y^{\vee\vee} \xrightarrow{1_S \otimes \text{ev}_{Y^\vee}^\tau} S \xrightarrow{\lambda_{[S], [X]}} S, \\
&\text{if } (Com)_{\lambda_{S^\vee, X^\vee}} \text{ is commutative.}
\end{aligned}$$

Suppose that $(Cas)_{\mu_{X, S}}$ is satisfied and that $(Com)_{\lambda_{[X], [S]}}$ is commutative. Then we have

$$\begin{aligned}
\mu_{X, S} : X^\vee &\xrightarrow{D_{X^\vee, S}} S \otimes Y^\vee \xrightarrow{D_{S, X^\vee} \otimes 1_{Y^\vee}} X^\vee \otimes Y^{\vee\vee} \otimes Y^\vee \xrightarrow{\text{ev}_{X^\vee}^\tau} X^\vee \xrightarrow{\lambda_{[X], [S]} \lambda_{S, X}} X^\vee, \\
&\text{if } (Com)_{\lambda_{S, X}} \text{ is commutative,} \\
\mu_{X, S} \lambda_{X, S} : X^\vee &\xrightarrow{D_{X^\vee, S}} S \otimes Y^\vee \xrightarrow{D_{S, X^\vee} \otimes 1_{Y^\vee}} X^\vee \otimes Y^{\vee\vee} \otimes Y^\vee \xrightarrow{\text{ev}_{X^\vee}^\tau} X^\vee \xrightarrow{\lambda_{[X], [S]}} X^\vee, \\
&\text{if } (Com)_{\lambda_{X, S}} \text{ is commutative.}
\end{aligned}$$

We have the similar statements exchanging the roles of S and X in the assumptions and the claims.

Proof. Suppose that $(Cas)_{\mu_{S,X}}$ is satisfied, that $(Com)_{\lambda_{[S],[X]}}$ is commutative and consider the following diagram:

$$\begin{array}{c}
\begin{array}{ccccccc}
S^\vee \otimes S & \xrightarrow{1_{S^\vee} \otimes D_{S,X^\vee}} & S^\vee \otimes X^\vee \otimes Y^{\vee\vee} & \xrightarrow{1_{S^\vee} \otimes D_{X^\vee,S} \otimes 1_{Y^{\vee\vee}}} & S^\vee \otimes S \otimes Y^\vee \otimes Y^{\vee\vee} & \xrightarrow{1_{S^\vee} \otimes S \otimes \text{ev}_{Y^\vee}^\tau} & S^\vee \otimes S \\
\downarrow \mu_{S,X} \cdot \text{ev}_S & \searrow D_{S^\vee,X} \otimes D_{S,X^\vee} & \downarrow D_{S^\vee,X} \otimes 1_{X^\vee \otimes Y^{\vee\vee}} & \searrow \lambda_{X^\vee,S^\vee} \cdot \text{ev}_S \otimes 1_{Y^\vee \otimes Y^{\vee\vee}} & \downarrow & \searrow & \\
X \otimes Y^\vee & \xrightarrow{\lambda_{[S],[X]} \cdot \text{ev}_{13,24}^{\tau,\tau}} & X^\vee \otimes Y^{\vee\vee} & \xrightarrow{\text{ev}_{13,Y^\vee}^\tau \otimes 1_{Y^{\vee\vee}}} & Y^\vee \otimes Y^{\vee\vee} & \xrightarrow{\lambda_{[S],[X]} \cdot \text{ev}_{Y^\vee}^\tau} & S^\vee \otimes S \\
\downarrow \lambda_{[S],[X]} \cdot \text{ev}_{13,24}^{\tau,\tau} & & \downarrow \lambda_{[S],[X]} \cdot \text{ev}_{Y^\vee}^\tau & & \downarrow \lambda_{[S],[X]} \cdot \text{ev}_{Y^\vee}^\tau & & \\
\mathbb{I} & & \mathbb{I} & & \mathbb{I} & & \mathbb{I}
\end{array}
\end{array}$$

(A) (B) (C)

Here the region (A) is commutative by Proposition 3.2, (B) by Lemma 3.3 when $(Com)_{\lambda_{X^\vee,S^\vee}}$ is commutative and (C) by definition of $\text{ev}_{13,24}^{\tau,\tau}$, $\text{ev}_{13,Y^\vee}^\tau$ and $\text{ev}_{Y^\vee}^\tau$. We deduce, setting $a := \lambda_{[S],[X]} \lambda_{X^\vee,S^\vee} \cdot (1_S \otimes \text{ev}_{Y^\vee}^\tau) \circ (D_{X^\vee,S} \otimes 1_{Y^{\vee\vee}}) \circ D_{S,X^\vee}$, the equality

$$\text{ev}_S \circ (1_{S^\vee} \otimes a) = \text{ev}_S \circ (1_{S^\vee} \otimes \mu_{S,X}).$$

Hence, by Remark 3.4, we get $a = \mu_{S,X}$. The commutativity of the other diagrams is proved in a similar way. \square

As an immediate consequence of Theorem 3.5 we get, in light of Remark 3.1, the following result.

Corollary 3.6. *Suppose that $(Cas)_{\mu_{S,X}}$ and $(Cas)_{\mu_{X,S}}$ are satisfied, that $(Com)_{\lambda_{S,X}}$, $(Com)_{\lambda_{S^\vee,X^\vee}}$, $(Com)_{\lambda_{X,S}}$ and $(Com)_{\lambda_{X^\vee,S^\vee}}$ are commutative, that $\mu_{S,X}$, $\mu_{X,S}$, $\lambda_{S,X}$, λ_{S^\vee,X^\vee} , $\lambda_{X,S}$ and λ_{X^\vee,S^\vee} are invertible and that Y is an invertible object. Then D_{S,X^\vee} , $D_{X^\vee,S}$, $D_{S^\vee,X}$, D_{X,S^\vee} , $f_{S,X}$, $f_{X,S}$, f_{S^\vee,X^\vee} and f_{X^\vee,S^\vee} are isomorphisms.*

Another important result for us will be the following corollary of Theorem 3.5. Define the following morphisms

$$\begin{aligned}
\varphi_{X^\vee,S^\vee}^{13} &: X^\vee \otimes Y^{\vee\vee} \otimes S^\vee \otimes Y^{\vee\vee} \xrightarrow{1_{X^\vee} \otimes \tau_{Y^{\vee\vee},S^\vee} \otimes 1_{Y^{\vee\vee}}} X^\vee \otimes S^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee} \\
&\quad \xrightarrow{\varphi_{X^\vee,S^\vee}^{13} \otimes 1_{Y^{\vee\vee} \otimes Y^{\vee\vee}}} Y^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee}, \\
\varphi_{X,S}^{13} &: X \otimes Y^\vee \otimes S \otimes Y^\vee \xrightarrow{1_X \otimes \tau_{Y^\vee,S} \otimes 1_{Y^\vee}} X \otimes S \otimes Y^\vee \otimes Y^\vee \xrightarrow{\varphi_{X,S}^{13} \otimes 1_{Y^\vee \otimes Y^\vee}} Y \otimes Y^\vee \otimes Y^\vee,
\end{aligned}$$

as well as

$$\begin{aligned}
\varphi_{X^\vee,S^\vee}^{13 \rightarrow Y^{\vee\vee}} &: X^\vee \otimes Y^{\vee\vee} \otimes S^\vee \otimes Y^{\vee\vee} \xrightarrow{\varphi_{X^\vee,S^\vee}^{13}} Y^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee} \xrightarrow{\text{ev}_{Y^\vee}^\tau \otimes 1_{Y^{\vee\vee}}} Y^{\vee\vee}, \\
\varphi_{X,S}^{13 \rightarrow Y^\vee} &: X \otimes Y^\vee \otimes S \otimes Y^\vee \xrightarrow{\varphi_{X,S}^{13}} Y \otimes Y^\vee \otimes Y^\vee \xrightarrow{\text{ev}_Y^\tau \otimes 1_{Y^\vee}} Y^\vee.
\end{aligned}$$

Corollary 3.7. *Suppose that $(Cas)_{\mu_{S,X}}$ is satisfied and that $(Com)_{\lambda_{[S],[X]}}$, $(Com)_{\lambda_{X,S}}$ and $(Com)_{\lambda_{X^\vee,S^\vee}}$ are commutative. Then, setting $\mu := \mu_{S,X}$ and $\lambda := \lambda_{[S],[X]} \lambda_{X^\vee,S^\vee} \lambda_{X,S}$, the following diagrams are commutative:*

$$\begin{array}{ccc}
S \otimes X & \xrightarrow{\varphi_{S,X}} & Y \\
\downarrow D_{S,X^\vee} \otimes D_{X,S^\vee} & & \downarrow \mu_{Y^\vee} \\
X^\vee \otimes Y^{\vee\vee} \otimes S^\vee \otimes Y^{\vee\vee} & \xrightarrow{\lambda \cdot \varphi_{X^\vee,S^\vee}^{13 \rightarrow Y^{\vee\vee}}} & Y^{\vee\vee}
\end{array}
\quad
\begin{array}{ccc}
S^\vee \otimes X^\vee & \xrightarrow{\varphi_{S^\vee,X^\vee}} & Y^\vee \\
\downarrow D_{S^\vee,X} \otimes D_{X^\vee,S} & & \downarrow \mu \\
X \otimes Y^\vee \otimes S \otimes Y^\vee & \xrightarrow{\lambda \cdot \varphi_{X,S}^{13 \rightarrow Y^\vee}} & Y^\vee
\end{array}$$

Proof. Suppose that $(Cas)_{\mu_{S,X}}$ is satisfied and that $(Com)_{\lambda_{[S],[X]}}$, $(Com)_{\lambda_{X,S}}$ and $(Com)_{\lambda_{X^\vee,S^\vee}}$ are commutative. Consider the following diagram:

$$\begin{array}{ccccc}
S \otimes X & \xrightarrow{D_{S,X^\vee} \otimes 1_X} & X^\vee \otimes Y^{\vee\vee} \otimes X & \xrightarrow{\mu_{S,X} \cdot \text{ev}_{13,Y^{\vee\vee}}^\phi} & Y^{\vee\vee} \\
\downarrow 1_S \otimes D_{X,S^\vee} & (A) & \downarrow \mu_{S,X} \lambda_{X,S} \cdot \text{ev}_S^\tau \otimes 1_{Y^{\vee\vee}} & (B) & \downarrow \lambda_{X,S} \cdot \text{ev}_S^\tau \otimes 1_{Y^{\vee\vee}} \\
S \otimes S^\vee \otimes Y^{\vee\vee} & \xrightarrow{\mu_{S,X}} & S \otimes S^\vee \otimes Y^{\vee\vee} & & S \otimes S^\vee \otimes Y^{\vee\vee} \\
\downarrow D_{S,X^\vee} \otimes 1_{S^\vee \otimes Y^{\vee\vee}} & (C) & \downarrow \lambda_{[S],[X]} \lambda_{X^\vee,S^\vee} \cdot 1_{S^\vee \otimes Y^{\vee\vee}} & (D) & \downarrow \lambda_{[S],[X]} \lambda_{X^\vee,S^\vee} \cdot \text{ev}_{Y^\vee}^\tau \otimes 1_{S^\vee \otimes Y^{\vee\vee}} \\
X^\vee \otimes Y^{\vee\vee} \otimes S^\vee \otimes Y^{\vee\vee} & \xrightarrow{D_{X^\vee,S} \otimes 1_{Y^{\vee\vee} \otimes S^\vee \otimes Y^{\vee\vee}}} & S \otimes Y^\vee \otimes Y^{\vee\vee} \otimes S^\vee \otimes Y^{\vee\vee} & & S \otimes Y^\vee \otimes Y^{\vee\vee} \otimes S^\vee \otimes Y^{\vee\vee} \\
\downarrow 1_{X^\vee} \otimes \tau_{Y^{\vee\vee},S^\vee} \otimes 1_{Y^{\vee\vee}} & (\tau) & \downarrow 1_{S \otimes Y^\vee} \otimes \tau_{Y^{\vee\vee},S^\vee} \otimes 1_{Y^{\vee\vee}} & (D) & \downarrow \text{ev}_{14}^\tau \\
X^\vee \otimes S^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee} & \xrightarrow{D_{X^\vee,S} \otimes 1_{S^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee}}} & S \otimes Y^\vee \otimes S^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee} & \xrightarrow{\text{ev}_{13}^\tau} & Y^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee}
\end{array}$$

The region (A) is commutative by Lemma 3.3, the commutativity of (B) is clear, (C) is commutative by Theorem 3.5 and (D) by definition of the evaluation maps (we have written $\text{ev}_{14}^\tau := \text{ev}_{14,Y^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee}}^\tau$ for shortness and similarly for ev_{13}^τ). Recalling that we have, by definition, $D_{S,X^\vee} = D_{\iota_{S,X}}$ and $D_{X^\vee,S} = D_{\iota_{X^\vee,S^\vee}}^*$, it follows from Lemma 2.7 we have $i_Y \circ \varphi_{S,X} = \text{ev}_{13,Y^{\vee\vee}}^\phi \circ (D_{\iota_{S,X}} \otimes 1_X)$ and $\varphi_{X^\vee,S^\vee} \otimes 1_{Y^{\vee\vee} \otimes Y^{\vee\vee}} = \text{ev}_{13,Y^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee}}^\tau \circ (D_{X^\vee,S} \otimes 1_{S^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee}})$. Hence the commutative diagram gives the claimed equality:

$$\begin{aligned}
\mu_{S,X} \cdot i_Y \circ \varphi_{S,X} &= \lambda_{[S],[X]} \lambda_{X^\vee,S^\vee} \lambda_{X,S} \cdot (\text{ev}_{Y^\vee}^\tau \otimes 1_{Y^{\vee\vee}}) \circ (\varphi_{X^\vee,S^\vee} \otimes 1_{Y^{\vee\vee} \otimes Y^{\vee\vee}}) \\
&\circ (1_{X^\vee} \otimes \tau_{Y^{\vee\vee},S^\vee} \otimes 1_{Y^{\vee\vee}}) \circ (D_{S,X^\vee} \otimes D_{X,S^\vee}) = \lambda_{[S],[X]} \lambda_{X^\vee,S^\vee} \lambda_{X,S} \cdot \varphi_{X^\vee,S^\vee}^{13 \rightarrow Y^{\vee\vee}} \circ (D_{S,X^\vee} \otimes D_{X,S^\vee}).
\end{aligned}$$

The commutativity of the other diagram is proved in a similar way. \square

Corollary 3.8. Suppose that $(Cas)_{\mu_{S,X}}$ is satisfied, that $(Com)_{\lambda_{[S],[X]}}$, $(Com)_{\lambda_{X,S}}$ and $(Com)_{\lambda_{X^\vee,S^\vee}}$ are commutative and that Y is invertible of rank r_Y (so that $r_Y \in \{\pm 1\}$). For every morphism $g : A \rightarrow \text{hom}(Y^\vee, B)$ and $h : C \rightarrow \text{hom}(Y, D)$ the following diagrams are commutative, where μ and λ are as in Corollary 3.7:

$$\begin{array}{ccc}
A \otimes S \otimes X & \xrightarrow{D_g \otimes \varphi_{S,X}} & B \otimes Y^{\vee\vee} \otimes Y \\
\downarrow 1_A \otimes D_{S,X^\vee} \otimes D_{X,S^\vee} & & \downarrow \mu \cdot 1_B \otimes Y^{\vee\vee} \otimes i_Y \\
A \otimes X^\vee \otimes Y^{\vee\vee} \otimes S^\vee \otimes Y^{\vee\vee} & & C \otimes X \otimes Y^\vee \otimes S \otimes Y^\vee \\
\downarrow 1_A \otimes \varphi_{X^\vee,S^\vee}^{13} & & \downarrow 1_C \otimes \varphi_{X^\vee,S^\vee}^{13} \\
A \otimes Y^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee} & \xrightarrow{\lambda_{r_Y} \cdot \varphi_g \otimes 1_{Y^{\vee\vee} \otimes Y^{\vee\vee}}} & B \otimes Y^{\vee\vee} \otimes Y^{\vee\vee} \\
\downarrow \lambda_{r_Y} \cdot \varphi_g \otimes 1_{Y^{\vee\vee} \otimes Y^{\vee\vee}} & & \downarrow \mu \\
A \otimes Y^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee} & \xrightarrow{\lambda_{r_Y} \cdot \varphi_h \otimes 1_{Y^\vee \otimes Y^{\vee\vee}}} & D \otimes Y^\vee \otimes Y^\vee
\end{array}$$

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
A \otimes S \otimes X & \xrightarrow{1_A \otimes \varphi_{S,X}} & A \otimes Y & \xrightarrow{D_g \otimes 1_Y} & B \otimes Y^{\vee\vee} \otimes Y \\
\downarrow 1_A \otimes D_{S,X^\vee} \otimes D_{X,S^\vee} & & \downarrow 1_A \otimes C_{Y^\vee} \otimes 1_Y & (C) & \downarrow \mu \cdot 1_B \otimes Y^{\vee\vee} \otimes i_Y \\
A \otimes X^\vee \otimes Y^{\vee\vee} \otimes S^\vee \otimes Y^{\vee\vee} & (A) & \downarrow \mu \cdot 1_A \otimes i_Y & & A \otimes Y^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee} \\
\downarrow 1_A \otimes \varphi_{X^\vee,S^\vee}^{13} & & \downarrow \lambda \cdot 1_A \otimes \text{ev}_{Y^\vee}^\tau \otimes 1_{Y^{\vee\vee}} & (B) & \downarrow \varphi_g \otimes 1_{Y^{\vee\vee} \otimes Y^{\vee\vee}} \\
A \otimes Y^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee} & \xrightarrow{\lambda \cdot 1_A \otimes \text{ev}_{Y^\vee}^\tau \otimes 1_{Y^{\vee\vee}}} & A \otimes Y^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee} & \xrightarrow{r_Y \cdot 1_A \otimes (\text{ev}_{Y^\vee}^\tau)^{-1} \otimes 1_{Y^{\vee\vee}}} & A \otimes Y^\vee \otimes Y^{\vee\vee} \otimes Y^{\vee\vee}
\end{array}$$

The region (A) is commutative by Corollary 3.7. The region (B) because $r_Y = r_{Y^\vee} := \text{ev}_{Y^\vee}^\tau \circ C_{Y^\vee}$, implying that $C_{Y^\vee} = r_Y \cdot (\text{ev}_{Y^\vee}^\tau)^{-1}$ and the functoriality of \otimes . Finally (C) is commutative by (27). The commutativity of the first diagram follows and the commutativity of the second diagram is proved in the same way. \square

4. APPLICATION TO Δ -GRADED ALGEBRAS IN \mathcal{C}

If Δ is a commutative integral⁷ semigroup, a Δ -graded algebra in \mathcal{C} is a family $A = (A_i, \varphi_{i,j}^A)_{i,j \in \Delta}$ where $\varphi_{i,j} = \varphi_{i,j}^A : A_i \otimes A_j \rightarrow A_{i+j}$ are morphisms making the following diagrams commutative:

$$\begin{array}{ccc} A_i \otimes A_j \otimes A_k & \xrightarrow{1_{A_i} \otimes \varphi_{j,k}} & A_i \otimes A_{j+k} \\ \varphi_{i,j} \otimes 1_{A_k} \downarrow & & \downarrow \varphi_{i,j+k} \\ A_{i+j} \otimes A_k & \xrightarrow{\varphi_{i+j,k}} & A_{i+j+k}. \end{array} \quad (29)$$

There is an obvious notion of morphisms of Δ -graded algebras and of direct sum decomposition, given component-wise. If $i, j \in \Delta$, we write $j \geq i$ to mean that $\exists (j-i) \in \Delta$ (unique because Δ is integral) such that $(j-i) + i = j$.

If $j \geq i$, we have $\varphi_{i,j-i} = \varphi_{f_{i,j-i}} : A_i \otimes A_{j-i} \rightarrow A_j$, where $f_{i,j-i} = f_{i,j-i}^A : A_i \rightarrow \text{hom}(A_{j-i}, A_j)$, so that we may consider the associated internal multiplication morphism

$$\iota_{i,j} := \iota_{f_{i,j-i}} : A_i \rightarrow \text{hom}(A_j^\vee, A_{j-i}^\vee) \text{ corresponding to } \varphi_{\iota_{i,j}} : A_i \otimes A_j^\vee \rightarrow A_{j-i}^\vee.$$

Suppose that, for every $i \in \Delta$, we have given a biproduct decomposition $A_i = A_i^+ \oplus A_i^-$ and write i_W^\pm , p_W^\pm and e_W^\pm as usual for $W = A_i$ or A_i^\vee and we also set $i_i^\pm = i_W^\pm$, $p_i^\pm = p_W^\pm$ and $e_i^\pm = e_W^\pm$ when $W = A_i$ and $p_{i,\vee}^\pm = p_W^\pm$ when $W = A_i^\vee$. We make a choice $\varepsilon : \Delta \rightarrow \{\pm\}$ of factors for every $i \in \Delta$ that we may assume, without loss of generality, to be given by the constant function $\varepsilon_i = +$.

Next we consider

$$\begin{aligned} f_{i,j}^+ : A_i^+ &\xrightarrow{i_i^+} A_i \xrightarrow{f_{i,j}} \text{hom}(A_j, A_{i+j}) \xrightarrow{p_{\text{hom}(A_j^+, A_{i+j}^+)}} \text{hom}(A_j^+, A_{i+j}^+), \\ \varphi_{i,j}^+ : A_i^+ \otimes A_j^+ &\xrightarrow{i_i^+ \otimes i_j^+} A_i \otimes A_j \xrightarrow{\varphi_{i,j}} A_{i+j} \xrightarrow{p_{i+j}^+} A_{i+j}^+ \end{aligned}$$

and, for $j \geq i$,

$$\begin{aligned} \iota_{i,j}^+ : A_i^+ &\xrightarrow{i_i^+} A_i \xrightarrow{\iota_{i,j}} \text{hom}(A_j^\vee, A_{j-i}^\vee) \xrightarrow{p_{\text{hom}(A_j^{+\vee}, A_{j-i}^{+\vee})}} \text{hom}(A_j^{+\vee}, A_{j-i}^{+\vee}) \text{ and} \\ \varphi_{\iota_{i,j}}^+ : A_i^+ \otimes A_j^+ &\xrightarrow{i_i^+ \otimes i_j^+} A_i \otimes A_j^\vee \xrightarrow{\varphi_{\iota_{i,j}}} A_{j-i}^\vee \xrightarrow{p_{j-i}^{+\vee}} A_{j-i}^{+\vee}. \end{aligned}$$

The following result is a restatement of Lemma 2.3 and Proposition 2.5.

Proposition 4.1. *The following diagrams are commutative.*

(1) When $j \geq i$

$$\begin{array}{ccc} A_i \otimes A_{j-i} \otimes A_j^\vee & \xrightarrow{(1_{A_{j-i}} \otimes \varphi_{\iota_{i,j}}) \circ (\tau_{A_i, A_{j-i}} \otimes 1_{A_j^\vee})} & A_{j-i} \otimes A_{j-i}^\vee \\ \varphi_{i,j-i} \otimes 1_{A_j^\vee} \downarrow & & \downarrow \text{ev}_{A_{j-i}}^\tau \\ A_j \otimes A_j^\vee & \xrightarrow{\text{ev}_{A_j}^\tau} & \mathbb{I}. \end{array}$$

(2) When $k \geq i$ and $k-i \geq j$,

$$\begin{array}{ccc} A_j \otimes A_i \otimes A_k^\vee & \xrightarrow{1_{A_j} \otimes \varphi_{\iota_{i,k}}} & A_j \otimes A_{k-i}^\vee \\ \varphi_{i,j}^\tau \otimes 1_{A_k^\vee} \downarrow & & \downarrow \varphi_{\iota_{j,k-i}} \\ A_{i+j} \otimes A_k^\vee & \xrightarrow{\varphi_{i+j,k}} & A_{k-i-j}^\vee \end{array} \quad \begin{array}{ccc} A_i \otimes A_j & \xrightarrow{\iota_{i,k} \otimes \iota_{j,k-i}} & \text{hom}(A_k^\vee, A_{k-i}^\vee) \otimes \text{hom}(A_{k-i}^\vee, A_{k-i-j}^\vee) \\ \varphi_{i,j} \downarrow & & \downarrow c_{A_k^\vee, A_{k-i}^\vee, A_{k-i-j}^\vee}^\tau \\ A_{i+j} & \xrightarrow{\iota_{i+j,k}} & \text{hom}(A_k^\vee, A_{k-i-j}^\vee), \end{array}$$

where

$$\varphi_{i,j}^\tau : A_j \otimes A_i \xrightarrow{\tau_{A_j, A_i}} A_i \otimes A_j \xrightarrow{\varphi_{i,j}} A_{i+j}.$$

⁷Integrality means that we may left (and right) simplify.

Suppose that $e_{i+j}^+ \circ \varphi_{i,j} \circ (e_i^+ \otimes 1_{A_j}) = e_{i+j}^+ \circ \varphi_{i,j}$ and $e_{i+j}^+ \circ \varphi_{i,j} \circ (1_{A_i} \otimes e_j^+) = e_{i+j}^+ \circ \varphi_{i,j}$. Then $A^+ := (A_i^+, \varphi_{i,j}^+)_{i,j \in \Delta}$ is a Δ -graded algebra in \mathcal{C} , we have $\varphi_{f_{i,j}^+} = \varphi_{i,j}^+$, the internal multiplication morphisms $\iota_{f_{i,j}^+}$ associated to this algebra structure are explicitly given by $\iota_{f_{i,j}^+} = \iota_{i,j}^+$ and $\varphi_{\iota_{f_{i,j}^+}} = \varphi_{\iota_{i,j}^+} = \varphi_{i,j}^+$ and we have the analogue of the above commutative diagrams with the $+$ sign inserted. Furthermore, we have a biproduct decomposition $A = A^+ \oplus A^-$ as Δ -graded algebras, where $A^- := (A_i^-, \varphi_{i,j}^-)_{i,j \in \Delta}$ satisfies the analogue results.

We will assume from now on that we have given Δ -graded algebras $A = (A_i, \varphi_{i,j}^A)_{i,j \in \Delta}$ and $A^\vee = (A_i^\vee, \varphi_{i,j}^{A^\vee})_{i,j \in \Delta}$ and \mathcal{C} is rigid. Then we define a $\Delta \times \Delta$ -graded family $A \otimes A^\vee := (A_j^i, \varphi_{j,l}^{i,k})_{(i,j),(k,l) \in \Delta \times \Delta}$ by the rule $A_j^i := A_i \otimes A_j^\vee$ and

$$\varphi_{j,l}^{i,k} := \varphi_{i,k}^A \otimes_\epsilon \varphi_{j,l}^{A^\vee} : A_j^i \otimes A_l^k \rightarrow A_{j+l}^{i+k}, \text{ associated to } f_{j,l}^{i,k} := f_{i,k}^A \otimes_\epsilon f_{j,l}^{A^\vee} : A_j^i \rightarrow \text{hom}(A_l^k, A_{j+l}^{i+k})$$

by Lemma 2.10. It is easily checked that $A \otimes A^\vee$ is indeed a $\Delta \times \Delta$ -graded algebra.

Next we define, when $l \geq i$ and $k \geq j$,

$$\begin{aligned} \delta_{i,l}^A &:= \varphi_{\iota_{i,l}} = \varphi_{\iota_{f_{i,l-i}^A}} : A_i \otimes A_l^\vee \rightarrow A_{l-i}^\vee, \\ \delta_{j,k}^{A^\vee} &:= \varphi_{\iota_{f_{j,k-j}^{A^\vee}}}^* : A_j^\vee \otimes A_k \rightarrow A_{k-j} \end{aligned}$$

and

$$\delta_{j,l}^{i,k} := \delta_{j,k}^{A^\vee} \otimes_\epsilon \delta_{i,l}^A : A_j^i \otimes A_l^k \rightarrow A_{l-i}^{k-j}, \text{ associated to } \iota_{j,l}^{i,k} := \iota_{f_{j,k-j}^{A^\vee}}^* \otimes_\epsilon \iota_{f_{i,l-i}^A} : A_j^i \rightarrow \text{hom}(A_l^k, A_{l-i}^{k-j})$$

by Lemma 2.10.

Applying Proposition 4.1 to $A \otimes A^\vee$ we find, thanks to Corollary 2.12 and (24), the following result.

Corollary 4.2. *The following diagrams are commutative.*

(1) When $l \geq i$ and $k \geq j$,

$$\begin{array}{ccc} A_j^i \otimes A_{k-j}^{l-i} \otimes A_l^k & \xrightarrow{(1_{A_{k-j}^{l-i}} \otimes \delta_{j,l}^{i,k}) \circ (\tau_{A_j^i, A_{k-j}^{l-i}} \otimes 1_{A_l^k})} & A_{k-j}^{l-i} \otimes A_{l-i}^{k-j} \\ \downarrow \varphi_{j,k-j}^{i,l-i} \otimes 1_{A_l^k} & & \downarrow \text{ev}_{14,23}^{\tau, \phi} \\ A_k^l \otimes A_l^k & \xrightarrow{\text{ev}_{14,23}^{\tau, \phi}} & \mathbb{I}. \end{array}$$

(2) When $n \geq i$, $m \geq j$, $n-i \geq k$ and $m-j \geq l$,

$$\begin{array}{ccc} A_l^k \otimes A_j^i \otimes A_n^m & \xrightarrow{1_{A_l^k} \otimes \delta_{j,n}^{i,m}} & A_l^k \otimes A_{n-i}^{m-j} \\ \downarrow \varphi_{j,l}^{i,k, \tau} \otimes 1_{A_n^m} & & \downarrow \delta_{l,n-i}^{k,m-j} \\ A_{j+l}^{i+k} \otimes A_n^m & \xrightarrow{\delta_{j+l,n}^{i+k,m}} & A_{n-i-k}^{m-j-l} \end{array} \quad \begin{array}{ccc} A_j^i \otimes A_l^k & \xrightarrow{\iota_{j,n}^{i,m} \otimes \iota_{l,n-i}^{k,m-j}} & \text{hom}(A_n^m, A_{n-i}^{m-j}) \otimes \text{hom}(A_{n-i}^{m-j}, A_{n-i-k}^{m-j-l}) \\ \downarrow \varphi_{j,l}^{i,k} & & \downarrow c_{A_n^m, A_{n-i}^{m-j}, A_{n-i-k}^{m-j-l}}^\tau \\ A_{j+l}^{i+k} & \xrightarrow{\iota_{j+l,n}^{i+k,m}} & \text{hom}(A_n^m, A_{n-i-k}^{m-j-l}), \end{array}$$

where

$$\varphi_{j,l}^{i,k, \tau} : A_l^k \otimes A_j^i \xrightarrow{\tau_{A_l^k, A_j^i}} A_j^i \otimes A_l^k \xrightarrow{\varphi_{j,l}^{i,k}} A_{j+l}^{i+k}.$$

We define, when $g \geq i$,

$$D^{i,g} := D_{\iota_{f_{i,g-i}^A}} : A_i \rightarrow A_{g-i}^\vee \otimes A_g^{\vee\vee} \text{ and } D_{i,g} := D_{\iota_{f_{i,g-i}^{A^\vee}}}^* : A_i^\vee \rightarrow A_{g-i} \otimes A_g^\vee.$$

We leave to the reader to restate the result of the previous section in this context.

4.1. Symmetric and alternating algebras. Suppose now that we have given a rigid and pseudo-abelian object $V \in \mathcal{C}$ and that \mathcal{C} is \mathbb{Q} -linear. The associativity constraint of \mathcal{C} implies that we may define an \mathbb{N} -graded tensor algebra $\otimes V$ by the rule $\varphi_{i,j}^{V,t} := 1_{\otimes^{i+j}V} : (\otimes^i V) \otimes (\otimes^j V) \rightarrow \otimes^{i+j}V$.

The permutation group S_i acts on $\otimes^i V$ and, for a character χ of S_i and a subset $S \subset S_i$, we define

$$e_S^\chi := \frac{1}{\#S} \sum_{\chi \in S_i} \chi^{-1}(\sigma) \sigma. \quad (30)$$

There are exactly two characters of S_i , namely $\chi = \varepsilon$ (the sign character) and $\chi = 1$ (the trivial character), which are distinct when $i \geq 2$. We let $\wedge^i V$ (resp. $\vee^i V$) the biproduct factor of $\otimes^i V$ which corresponds to the idempotent $e_{a,V}^i := e_{S_i}^\varepsilon$ (resp. $e_{s,V}^i := e_{S_i}^1$), which exists because we assume that V is pseudo-abelian. We write $i_{V,a}^i : \wedge^i V \rightarrow \otimes^i V$ (resp. $i_{V,s}^i : \vee^i V \rightarrow \otimes^i V$) and $p_{V,a}^i : \otimes^i V \rightarrow \wedge^i V$ (resp. $p_{V,s}^i : \otimes^i V \rightarrow \vee^i V$) for the associated injective and surjective morphisms. Next we claim that, setting

$$\begin{aligned} \varphi_{i,j}^{V,a} &: \wedge^i V \otimes \wedge^j V \xrightarrow{i_{V,a}^i \otimes i_{V,a}^j} (\otimes^i V) \otimes (\otimes^j V) \xrightarrow{\varphi_{i,j}^{V,t}} \otimes^{i+j} V \xrightarrow{p_{V,a}^{i+j}} \wedge^{i+j} V, \\ \varphi_{i,j}^{V,s} &: \vee^i V \otimes \vee^j V \xrightarrow{i_{V,s}^i \otimes i_{V,s}^j} (\otimes^i V) \otimes (\otimes^j V) \xrightarrow{\varphi_{i,j}^{V,t}} \otimes^{i+j} V \xrightarrow{p_{V,s}^{i+j}} \vee^{i+j} V \end{aligned}$$

give rise to \mathbb{N} -graded algebras, called respectively the alternating and the symmetric algebras on V . Since $\varphi_{i,j}^{V,t} = 1_{\otimes^{i+j}V}$ we may check, according to Proposition 4.1, that we have $e_{S_{i+j}}^\chi \circ (e_{S_i}^\chi \otimes 1_{\otimes^j V}) = e_{S_{i+j}}^\chi$ and $e_{S_{i+j}}^\chi \circ (1_{\otimes^i V} \otimes e_{S_j}^\chi) = e_{S_{i+j}}^\chi$ for $\chi = \varepsilon$ or $\chi = 1$. But we have that the action of $e_{S_i}^\chi \otimes 1_{\otimes^j V} \in \text{End}(\otimes^{i+j} V)$ (resp. $1_{\otimes^i V} \otimes e_{S_j}^\chi \in \text{End}(\otimes^{i+j} V)$) is obtained by identifying $S_i \simeq S_{\{1,\dots,i\}} \subset S_{i+j}$ (resp. $S_j \simeq S_{\{i+1,\dots,i+j\}} \subset S_{i+j}$) and it is given by $e_{S_{\{1,\dots,i\}}}^\chi \in \mathbb{Q}[S_{i+j}]$ (resp. $e_{S_{\{i+1,\dots,i+j\}}}^\chi \in \mathbb{Q}[S_{i+j}]$). Hence the claimed relation follows from the identity $e_{S_{i+j}}^\chi e_{S_{\{1,\dots,i\}}}^\chi = e_{S_{i+j}}^\chi$ (resp. $e_{S_{i+j}}^\chi e_{S_{\{i+1,\dots,i+j\}}}^\chi = e_{S_{i+j}}^\chi$) taking place in $\mathbb{Q}[S_{i+j}]$.

We note that it follows from the definitions that $\varphi_{i,j}^{V,a}$ and $\varphi_{i,j}^{V,s}$ are uniquely characterized by the property of making the following diagrams commutative:

$$\begin{array}{ccc} (\otimes^i V) \otimes (\otimes^j V) & \xrightarrow{1_{\otimes^{i+j}V}} & \otimes^{i+j} V \\ \downarrow p_{V,a}^i \otimes p_{V,a}^j & & \downarrow p_{V,a}^{i+j} \\ \wedge^i V \otimes \wedge^j V & \xrightarrow{\varphi_{i,j}^{V,a}} & \wedge^{i+j} V, \end{array} \quad \begin{array}{ccc} (\otimes^i V) \otimes (\otimes^j V) & \xrightarrow{1_{\otimes^{i+j}V}} & \otimes^{i+j} V \\ \downarrow p_{V,s}^i \otimes p_{V,s}^j & & \downarrow p_{V,s}^{i+j} \\ \vee^i V \otimes \vee^j V & \xrightarrow{\varphi_{i,j}^{V,s}} & \vee^{i+j} V. \end{array} \quad (31)$$

Next we remark that we may use rigidity of V to canonically identify $\epsilon_V^i : (\otimes^i V^\vee, \text{ev}_V^i) \rightarrow ((\otimes^i V)^\vee, \text{ev}_{\otimes^i V})$: in other words $(\otimes^i V^\vee, \text{ev}_V^i)$ is a dual pair for $\otimes^i V$. Next, we define

$$\begin{aligned} \text{ev}_{V,a}^i &: \wedge^i V^\vee \otimes \wedge^i V \xrightarrow{i_{V^\vee,a}^i \otimes i_{V,a}^i} (\otimes^i V^\vee) \otimes (\otimes^i V) \xrightarrow{\text{ev}_V^i} \mathbb{I}, \\ \text{ev}_{V,s}^i &: \vee^i V^\vee \otimes \vee^i V \xrightarrow{i_{V^\vee,s}^i \otimes i_{V,s}^i} (\otimes^i V^\vee) \otimes (\otimes^i V) \xrightarrow{\text{ev}_V^i} \mathbb{I} \end{aligned}$$

and we claim that $(\wedge^i V^\vee, \text{ev}_{V,a}^i)$ is a dual pair for $\wedge^i V$ and $(\vee^i V^\vee, \text{ev}_{V,s}^i)$ is a dual pair for $\vee^i V$. Indeed we have $\text{ev}_V^i \circ (\sigma \otimes \sigma) = \text{ev}_V^i$ for every $\sigma \in S_i$, because the canonical morphism $\otimes^i \mathbb{I} \rightarrow \mathbb{I}$ appearing in the definition of ev_V^i is S_i -invariant. Equivalently, $\text{ev}_V^i \circ (\sigma^{-1} \otimes 1_{\otimes^i V}) = \text{ev}_V^i \circ (1_{\otimes^i V^\vee} \otimes \sigma)$ proving that $\sigma^{-1} = \sigma^\vee$ from which it follows that $(e_{S_i}^\chi)^\vee = e_{S_i}^\chi$. Then our claim follows from (10) (with $Y = Y^\pm = \mathbb{I}$).

5. A POINCARÉ DUALITY ISOMORPHISM FOR THE ALTERNATING ALGEBRAS

In this section we suppose that \mathcal{C} is rigid, \mathbb{Q} -linear and pseudo-abelian. We consider an object $V \in \mathcal{C}$ and we apply the results on Δ -graded algebras with $A = (\wedge \cdot V, \varphi_{i,j}^{V,a})$ and $A^\vee = (\wedge \cdot V^\vee, \varphi_{i,j}^{V^\vee,a})$. We will use the shorter notations $i_V^p := i_{V,a}^p$, $p_V^p := p_{V,a}^p$, $e_V^p := e_{V,a}^p$, $\varphi_{i,j} := \varphi_{i,j}^{V,a} := \varphi_{i,j}^{V,a}$ and $\varphi_{i,j}^{V^\vee} := \varphi_{i,j}^{V^\vee,a}$. In order to make explicit the internal multiplication morphisms we define, for every $j \geq i$,

$$\varphi_{i,j}^{V,t} := \text{ev}_V^{i,\tau} \otimes 1_{\otimes^{j-i} V^\vee} : (\otimes^i V) \otimes (\otimes^j V^\vee) \rightarrow \otimes^{j-i} V^\vee.$$

It is readily checked that $\varphi_{\iota_{i,j}^{V,t}}$ satisfies the characterizing property (15) of Proposition 2.2 with $\varphi_f = \varphi_{i,j-i}^{V,t} = 1_{\otimes^j V}$. Since the alternating algebra is obtained from the tensor algebra as in Proposition 4.1, it follows that the internal multiplication $\iota_{i,j} = \iota_{i,j}^{V,a} := \iota_{i,j}^A$ of the alternating algebra is explicitly given by

$$\varphi_{\iota_{i,j}} : \wedge^i V \otimes \wedge^j V^\vee \xrightarrow{\iota_V^i \otimes \iota_{V^\vee}^j} (\otimes^i V) \otimes (\otimes^j V^\vee) \xrightarrow{\text{ev}_V^{i,\tau} \otimes 1_{\otimes^{j-i} V^\vee}} \otimes^{j-i} V^\vee \xrightarrow{p_{V^\vee}^{j-i}} \wedge^{j-i} V^\vee.$$

In order to make this morphism completely explicit, we note that $S_i \times S_j$ acts on $(\otimes^i V) \otimes (\otimes^j V^\vee)$ and we may identify $S_{j-i} \simeq S_{\{i+1, \dots, j\}} \subset S_j$ acting on $(\otimes^i V) \otimes (\otimes^j V^\vee)$. With this identification,

$$\varphi_{\iota_{i,j}^{V,t}} \circ \sigma = \sigma \circ \varphi_{\iota_{i,j}^{V,t}} \text{ for every } \sigma \in S_{j-i}. \quad (32)$$

Let

$$\mathcal{P}^{i \leq j} := \mathcal{P}^{\{i+1, \dots, j\}, \{1, \dots, j\}} = \left\{ p = (p_1, \dots, p_i) \in \{1, \dots, j\}^i : p_k \neq p_l \text{ for every } k \neq l \right\},$$

and, for every $p = (p_1, \dots, p_i) \in \mathcal{P}^{i \leq j}$, write $\delta_p^{i \leq j} \in S_j$ for a fixed permutation such that $\delta_p^{i \leq j}(p_k) = k$ for $k \in \{1, \dots, i\}$. Then $\{\delta_p^{i \leq j} : p \in \mathcal{P}^{i \leq j}\}$ is a set of coset representatives for $S_{\{i+1, \dots, j\}} \backslash S_j$ and, hence, $R := \{\delta \delta_p^{i \leq j} : p \in \mathcal{P}^{i \leq j}, \delta \in S_i\}$ is a set of coset representatives for $S_{\{i+1, \dots, j\}} \backslash S_i \times S_j$. We have, setting $e^{i \leq j} := \frac{(j-i)!}{j!} \sum_{p \in \mathcal{P}^{i \leq j}} \varepsilon^{-1} \left(\delta_p^{i \leq j} \right) \delta_p^{i \leq j}$,

$$\frac{(j-i)!}{j!i!} \sum_{\delta \in S_i, p \in \mathcal{P}^{i \leq j}} \varepsilon^{-1} \left(\delta \delta_p^{i \leq j} \right) \delta \delta_p^{i \leq j} = e_{S_i}^\varepsilon \cdot e^{i \leq j}, \quad (33)$$

which acts on $(\otimes^i V) \otimes (\otimes^j V^\vee)$ via $e_V^i \otimes e^{i \leq j}$. In particular, when $i = 1$, we have $\mathcal{P}^{1 \leq j} = \{1, \dots, j\}$ and $\delta_p^{1 \leq j}$ is any fixed permutation such that $\delta_p^{1 \leq j}(p) = 1$; we may take, for example, $\delta_p^{1 \leq j} = (1, \dots, p) =: c_p$ in this case and then $\varepsilon^{-1} \left(\delta_p^{1 \leq j} \right) = (-1)^{p-1}$. We can now prove the following result.

Lemma 5.1. *Setting*

$$\tilde{\varphi}_{\iota_{i,j}} := \left(\text{ev}_V^{i,\tau} \otimes 1_{\otimes^{j-i} V^\vee} \right) \circ \left(e_V^i \otimes e^{i \leq j} \right) = \frac{(j-i)!}{j!} \sum_{p \in \mathcal{P}^{i \leq j}} \varepsilon^{-1} \left(\delta_p^{i \leq j} \right) \cdot \left(\text{ev}_V^{i,\tau} \otimes 1_{\otimes^{j-i} V^\vee} \right) \circ \left(e_V^i \otimes \delta_p^{i \leq j} \right)$$

we have that $\varphi_{\iota_{i,j}}$ is the unique morphism making the following diagram commutative:

$$\begin{array}{ccc} (\otimes^i V) \otimes (\otimes^j V^\vee) & \xrightarrow{\tilde{\varphi}_{\iota_{i,j}}} & \otimes^{j-i} V^\vee \\ p_V^i \otimes p_{V^\vee}^j \downarrow & & \downarrow p_{V^\vee}^j \\ \wedge^i V \otimes \wedge^j V^\vee & \xrightarrow{\varphi_{\iota_{i,j}}} & \wedge^{j-i} V^\vee. \end{array}$$

In particular, when $i = 1$, setting $\text{ev}_{V,p}^\tau := (\text{ev}_V^\tau \otimes 1_{\otimes^{j-1} V^\vee}) \circ c_p$ ⁸, we may take

$$\tilde{\varphi}_{\iota_{1,j}} = \frac{1}{j} \sum_{p=1}^j (-1)^{p-1} \text{ev}_{0,p}.$$

Proof. Suppose that we have given a subgroup $H \subset G$ of a finite group G , that G acts on X , H acts on Y and that we have given $f : X \rightarrow Y$ which is H -equivariant. If χ is a character of G and $R_{H \backslash G}$ is a set of coset representative for $H \backslash G$, we may consider the elements e_G^χ , e_H^χ and $e_{R_{H \backslash G}}^\chi$ defined as in (30) with $S = G$, H or $R_{H \backslash G}$ and S_i replaced by a more general group G . We have that e_G^χ and e_H^χ are idempotents and we let $p_X^\chi : X \rightarrow X^\chi$ and $p_Y^\chi : Y \rightarrow Y^\chi$ be the associated surjective morphisms and i_X^χ the associated injective morphism. Then it is a general fact that, setting $f_{R_{H \backslash G}} := f \circ e_{R_{H \backslash G}}^\chi$ and $f^\chi := p_Y^\chi \circ f \circ i_X^\chi$, the morphism f^χ is characterized as the unique morphism such that $f^\chi \circ p_X^\chi = p_Y^\chi \circ f_{R_{H \backslash G}}$.

We may apply this general remark with $X = (\otimes^i V) \otimes (\otimes^j V^\vee)$, $Y = \otimes^{j-i} V^\vee$, $f = \text{ev}_V^{i,\tau} \otimes 1_{\otimes^{j-i} V^\vee}$, $H = S_{\{i+1, \dots, j\}}$ and $G = S_i \times S_j$; since f is H -equivariant by (32) and R is a set of coset representatives

⁸We have, symbolically,

$$\text{ev}_{0,p}(f \otimes x_1 \otimes \dots \otimes x_p \otimes \dots \otimes x_j) = \langle f, x_p \rangle x_1 \otimes \dots \otimes \hat{x}_p \otimes \dots \otimes x_j$$

for $S_{\{i+1,\dots,j\}} \setminus S_i \times S_j$, we deduce that $f^\varepsilon = \varphi_{\iota_{i,j}}$ is the unique morphism such that $f^\chi \circ p_X^\chi = p_Y^\chi \circ f_{R_{H \setminus G}}$, where thanks to (33) and the equality $\#R = \frac{(j-i)!}{j!i!}$,

$$f_{R_{H \setminus G}} = \left(\text{ev}_V^{i,\tau} \otimes 1_{\otimes^{j-i} V^\vee} \right) \circ (e_V^i \otimes e^{i \leq j}).$$

□

Beside the properties encoded in Proposition 4.1, the internal multiplication morphisms $\iota_{1,j}$ has another key property. In symbols it says that the normalized family $\iota_j := j \cdot \iota_{1,j}$ is an antiderivation, i.e.

$$\iota_{j+l}(x)(\omega_j \wedge \omega_l) = \iota_{j+l}(x)(\omega_j) + (-1)^j \iota_{j+l}(x)(\omega_l) \text{ for } x \in V, \omega_j \in \wedge^j V^\vee \text{ and } \omega_l \in \wedge^l V^\vee.$$

This is the content of the following proposition whose proof, based on Lemma 5.1, we leave to the reader.

Proposition 5.2. *The following diagram is commutative, when $j, l \geq 1$:*

$$\begin{array}{ccc} V \otimes \wedge^j V^\vee \otimes \wedge^l V^\vee & \xrightarrow{\left(j \cdot \varphi_{\iota_{1,j}} \otimes 1_{\wedge^l V^\vee}, (-1)^j l \cdot \left(1_{\wedge^j V^\vee} \otimes \varphi_{\iota_{1,l}} \right) \circ \left(\tau_{V, \wedge^j V^\vee} \otimes 1_{\wedge^l V^\vee} \right) \right)} & \wedge^{j-1} V^\vee \otimes \wedge^l V^\vee \oplus \wedge^j V^\vee \otimes \wedge^{l-1} V^\vee \\ \downarrow 1_V \otimes \varphi_{j,l} & & \downarrow \varphi_{j-1,l} \oplus \varphi_{j,l-1} \\ V \otimes \wedge^{j+l} V^\vee & \xrightarrow{(j+l) \varphi_{\iota_{1,j+l}}} & \wedge^{j+l-1} V^\vee. \end{array}$$

Working with the dual algebras one easily sees that, setting

$$\varphi_{\iota_{i,j}^{V^\vee}, t, *} := \text{ev}_V^i \otimes 1_{\otimes^{j-i} V} : (\otimes^i V^\vee) \otimes (\otimes^j V) \rightarrow \otimes^{j-i} V,$$

the morphism $\varphi_{\iota_{i,j}^{V^\vee}, t, *}$ satisfies the property (23) with $\varphi_g = \varphi_{i,j-i}^{V^\vee, t} = 1_{\otimes^j V}$, which is of course characterizing.

It follows that $\iota_{i,j}^* = \iota_{i,j}^{V^\vee, a, *} := \iota_{i,j}^{A^\vee, *}$ is obtained in the analogous way as $\iota_{i,j}$ was obtained and the analogous of Proposition 5.2 is true.

Exactly as we did with more general Δ -graded algebras, we can now define, when $l \geq i$ and $k \geq j$, $\delta_{i,l}^A := \varphi_{\iota_{i,l}} : \wedge^i V \otimes \wedge^l V^\vee \rightarrow \wedge^{l-i} V^\vee$, $\delta_{j,k}^{A^\vee} := \varphi_{\iota_{j,k}^*} : \wedge^j V^\vee \otimes \wedge^k V \rightarrow \wedge^{k-j} V^\vee$ and

$$\delta_{j,l}^{i,k} := \delta_{j,k}^{A^\vee} \otimes_\epsilon \delta_{i,l}^A : \wedge_j^i V \otimes \wedge_l^k V \rightarrow \wedge_{l-i}^{k-j} V, \text{ associated to } \iota_{j,l}^{i,k} := \iota_{j,k}^* \otimes_\epsilon \iota_{i,l} : \wedge_j^i V \rightarrow \text{hom} \left(\wedge_l^k V, \wedge_{l-i}^{k-j} V \right),$$

where $\wedge_q^p V := \wedge^p V \otimes \wedge^q V^\vee$. Beside the properties encoded in Corollary 4.2, the following property is enjoyed by the families $\delta_{0,q}^{1,p}$ and $\delta_{1,q}^{0,p}$: the proof is just an application of Proposition 5.2 and its dual statement for the second diagram.

Corollary 5.3. *If $j, l \geq 1$ then the following diagram is commutative:*

$$\begin{array}{ccc} \wedge_0^1 V \otimes \wedge_j^i V \otimes \wedge_l^k V & \xrightarrow{\left(j \cdot \delta_{0,j}^{1,i} \otimes 1_{\wedge_l^k V}, (-1)^j l \cdot \left(1_{\wedge_j^i V} \otimes \delta_{0,l}^{1,k} \right) \circ \left(\tau_{\wedge_0^1 V, \wedge_j^i V} \otimes 1_{\wedge_l^k V} \right) \right)} & \wedge_{j-1}^i V \otimes \wedge_l^k V \oplus \wedge_j^i V \otimes \wedge_{l-1}^k V \\ \downarrow 1_{\wedge_0^1 V} \otimes \varphi_{j,l}^{i,k} & & \downarrow \varphi_{j-1,l}^{i,k} \oplus \varphi_{j,l-1}^{i,k} \\ \wedge_0^1 V \otimes \wedge_{j+l}^{i+k} V & \xrightarrow{(j+l) \cdot \delta_{0,j+l}^{1,i+k}} & \wedge_{j+l-1}^{i+k} V \end{array}$$

The following diagram is commutative, when $i, k \geq 1$:

$$\begin{array}{ccc} \wedge_1^0 V \otimes \wedge_j^i V \otimes \wedge_l^k V & \xrightarrow{\left(i \cdot \delta_{1,j}^{0,i} \otimes 1_{\wedge_l^k V}, (-1)^i k \cdot \left(1_{\wedge_j^i V} \otimes \delta_{1,l}^{0,k} \right) \circ \left(\tau_{\wedge_1^0 V, \wedge_j^i V} \otimes 1_{\wedge_l^k V} \right) \right)} & \wedge_j^{i-1} V \otimes \wedge_l^k V \oplus \wedge_j^i V \otimes \wedge_{l-1}^{k-1} V \\ \downarrow 1_{\wedge_1^0 V} \otimes \varphi_{j,l}^{i,k} & & \downarrow \varphi_{j,l}^{i-1,k} \oplus \varphi_{j,l}^{i,k-1} \\ \wedge_1^0 V \otimes \wedge_{j+l}^{i+k} V & \xrightarrow{(i+k) \cdot \delta_{1,j+l}^{0,i+k}} & \wedge_{j+l}^{i+k-1} V. \end{array}$$

The proof of the following lemma, which is postponed to the subsequent subsection, is based on Corollaries 4.2 and 5.3.

Lemma 5.4. Let $r := \text{rank}(V)$ be the rank of V , defined as the composition

$$r : \mathbb{I} \xrightarrow{C_Y} V \otimes V^\vee \xrightarrow{\text{ev}_Y^\tau} \mathbb{I}.$$

For every $g \geq i$ we have the equality

$$\binom{g}{i}^{-1} \binom{r+i-g}{i} \cdot C_{\wedge^{g-i}V} = \delta_{i,g}^{i,g} \circ (C_{\wedge^i V} \otimes C_{\wedge^g V}),$$

where, for every $k \in \mathbb{N}_{\geq 1}$,

$$\binom{T}{k} := \frac{1}{k!} T(T-1) \dots (T-k+1) \in \mathbb{Q}[T] \text{ and } \binom{T}{0} = 1.$$

We define, when $g \geq i$,

$$D^{i,g} := D_{i,g} : \wedge^i V \rightarrow \wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee} \text{ and } D_{i,g} := D_{i,g}^* : \wedge^i V^\vee \rightarrow \wedge^{g-i} V \otimes \wedge^g V^\vee.$$

Thanks to Lemma 5.4, the commutative diagrams of Proposition 3.2, Lemma 3.3, Theorem 3.5 and, respectively, Corollary 3.7, translate into the following result.

Theorem 5.5. The following diagrams are commutative, for every $g \geq i \geq 0$.

(1)

$$\begin{array}{ccc} \wedge^i V \otimes \wedge^i V^\vee & \xrightarrow{\binom{g}{g-i}^{-1} \binom{r-i}{g-i} \text{ev}_{V,a}^{i,\tau}} & \mathbb{I} \\ \downarrow D^{i,g} \otimes D_{i,g} & \searrow \text{ev}_{13,24}^{\phi,\phi} & \\ \wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^{g-i} V \otimes \wedge^g V^\vee & \xrightarrow{\text{ev}_{13,24}^{\phi,\phi}} & \mathbb{I} \end{array}$$

(2)

$$\begin{array}{ccc} \wedge^i V^\vee \otimes \wedge^{g-i} V^\vee \xrightarrow{1_{\wedge^i V^\vee} \otimes D_{g-i,g}} \wedge^i V^\vee \otimes \wedge^{g-i} V \otimes \wedge^g V^\vee & & \wedge^i V \otimes \wedge^{g-i} V \xrightarrow{1_{\wedge^i V} \otimes D_{g-i,g}} \wedge^i V \otimes \wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee} \\ \downarrow D_{i,g} \otimes 1_{\wedge^{g-i} V^\vee} & \downarrow (-1)^{i(g-i)} \cdot \text{ev}_{V,a}^i \otimes 1_{\wedge^g V^\vee} & \downarrow D_{i,g} \otimes 1_{\wedge^{g-i} V} \\ \wedge^{g-i} V \otimes \wedge^g V^\vee \otimes \wedge^{g-i} V^\vee \xrightarrow{\text{ev}_{13,\wedge^g V^\vee}^\tau} \wedge^g V^\vee & & \wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^{g-i} V \xrightarrow{\text{ev}_{13,\wedge^g V^\vee}^\phi} \wedge^g V^{\vee\vee} \\ & & \downarrow (-1)^{i(g-i)} \cdot \text{ev}_{V,a}^{i,\tau} \otimes 1_{\wedge^g V^{\vee\vee}} \end{array}$$

(3)

$$\begin{array}{ccccc} & & (-1)^{i(g-i)} \binom{g}{g-i}^{-1} \binom{r-i}{g-i} & & \\ & \nearrow & & \searrow & \\ \wedge^i V & \xrightarrow{D^{i,g}} & \wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee} & \xrightarrow{D_{g-i,g} \otimes 1_{\wedge^g V^{\vee\vee}}} & \wedge^i V \otimes \wedge^g V^\vee \otimes \wedge^g V^{\vee\vee} & \xrightarrow{1_{\wedge^i V} \otimes \text{ev}_{V^\vee,a}^{g,\tau}} & \wedge^i V \end{array}$$

and

$$\begin{array}{ccccc} & & (-1)^{i(g-i)} \binom{g}{i}^{-1} \binom{r+i-g}{i} & & \\ & \nearrow & & \searrow & \\ \wedge^{g-i} V^\vee & \xrightarrow{D_{g-i,g}} & \wedge^i V \otimes \wedge^g V^\vee & \xrightarrow{D^{i,g} \otimes 1_{\wedge^g V^\vee}} & \wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^g V^\vee & \xrightarrow{1_{\wedge^{g-i} V^\vee} \otimes \text{ev}_{V^\vee,a}^{g,\tau}} & \wedge^{g-i} V^\vee \end{array}$$

(4)

$$\begin{array}{ccc} \wedge^i V \otimes \wedge^{g-i} V \xrightarrow{\varphi_{i,g-i}} \wedge^g V & & \wedge^i V^\vee \otimes \wedge^{g-i} V^\vee \xrightarrow{\varphi_{i,g-i}} \wedge^g V^\vee \\ \downarrow D^{i,g} \otimes D_{g-i,g} & \downarrow \binom{g}{g-i}^{-1} \binom{r-i}{g-i} \cdot i_{\wedge^g V} & \downarrow D_{i,g} \otimes D_{g-i,g} \\ \wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^i V^\vee \otimes \wedge^g V^{\vee\vee} \xrightarrow{\varphi_{13 \rightarrow \wedge^g V^{\vee\vee}}^{g-i,i}} \wedge^g V^{\vee\vee} & & \wedge^{g-i} V \otimes \wedge^g V^\vee \otimes \wedge^i V \otimes \wedge^g V^{\vee\vee} \xrightarrow{\varphi_{13 \rightarrow \wedge^g V^{\vee\vee}}^{g-i,i}} \wedge^g V^\vee \\ & & \downarrow \binom{g}{g-i}^{-1} \binom{r-i}{g-i} \end{array}$$

Proof. The commutative diagrams (1), (2), (3) and (4) are just Proposition 3.2, Lemma 3.3, Theorem 3.5 and, respectively, Corollary 3.7 with $(S, X, Y) = (\wedge^i V, \wedge^{g-i} V, \wedge^g V)$, $\varphi_{S,X} = \varphi_{i,g-i}^V$, $\varphi_{X,S} = \varphi_{g-i,i}^V$, $\varphi_{S^\vee, X^\vee} = \varphi_{i,g-i}^{V^\vee}$, $\varphi_{X^\vee, S^\vee} = \varphi_{g-i,i}^{V^\vee}$, $D_{S,X^\vee} = D^{i,g}$, $D_{X,S^\vee} = D^{g-i,g}$, $D_{S^\vee, X} = D_{i,g}$, $D_{X^\vee, S} = D_{g-i,g}$. Indeed we have in this case $\mu_{S,X} = \mu_{g-i,g} = \binom{g}{g-i}^{-1} \binom{r-i}{g-i}$, $\mu_{X,S} = \mu_{i,g} = \binom{g}{i}^{-1} \binom{r+i-g}{i}$, $\lambda_{S,X} = \lambda_{X,S} = \lambda_{S^\vee, X^\vee} = \lambda_{X^\vee, S^\vee} = (-1)^{i(g-i)}$ and $\lambda_{[X],[S]} = \lambda_{[S],[X]} = 1$. \square

We say that V has *alternating rank* $g \in \mathbb{N}_{\geq 1}$ if $\wedge^g V$ is an invertible object and $\binom{r-i}{g-i}$ and $\binom{r+i-g}{i}$ are invertible for every $0 \leq i \leq g$. For example, when $\text{End}(\mathbb{I})$ is a field or $r \in \mathbb{Q}$, the second condition means that r is not a root of the polynomials $\binom{r-i}{g-i} \in \mathbb{Q}[T]$ and $\binom{r+i-g}{i} \in \mathbb{Q}[T]$ for every $0 \leq i \leq g$, i.e. that $r \neq i, i+1, \dots, g-1$ and $r \neq g-i, g-i+1, \dots, g-1$ for every $1 \leq i \leq g$.

We say that V has *strong alternating rank* $g \in \mathbb{N}_{\geq 1}$ if $\wedge^g V$ is an invertible object and $r = g$ (hence V has alternating rank g). With these notations Corollary 3.6 specializes to the following result.

Corollary 5.6. *If V has weakly geometric rank $g \in \mathbb{N}$ then, for every $0 \leq i \leq g$, the morphisms $D^{i,g}$, $D_{g-i,g}$, $D^{g-i,g}$ and $D_{i,g}$ are isomorphisms and the multiplication maps $\varphi_{i,g-i}^V$, $\varphi_{g-i,i}^V$, $\varphi_{i,g-i}^{V^\vee}$ and $\varphi_{g-i,i}^{V^\vee}$ are perfect pairings (meaning that the associate hom valued morphisms are isomorphisms). Furthermore, when V has geometric rank g , we have $\binom{r-i}{g-i} = \binom{r+i-g}{i} = 1$ in the commutative diagrams of Theorem 5.5.*

We end this section with the following result.

Proposition 5.7. *The following diagrams are commutative, when $\wedge^g V$ is invertible of rank $r_{\wedge^g V}$ (hence $r_{\wedge^g V} \in \{\pm 1\}$):*

$$\begin{array}{ccc}
\wedge^i V \otimes \wedge^{g-i} V \otimes V & \xrightarrow{\tau_{\wedge^i V \otimes \wedge^{g-i} V, V}} & V \otimes \wedge^i V \otimes \wedge^{g-i} V \\
\downarrow (1_{\wedge^i V} \otimes \varphi_{g-i,1}, (1_{\wedge^{g-i} V} \otimes \varphi_{i,1}) \circ (\tau_{\wedge^i V, \wedge^{g-i} V} \otimes 1_V)) & & \downarrow D^{1,g} \otimes \varphi_{i,g-i} \\
\wedge^i V \otimes \wedge^{g-i+1} V \oplus \wedge^{g-i} V \otimes \wedge^{i+1} V & & \wedge^{g-1} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^g V \\
\downarrow D^{i,g} \otimes D^{g-i+1,g} \oplus D^{g-i,g} \otimes D^{i+1,g} & & \downarrow r_{\wedge^g V} g \binom{g}{g-i}^{-1} \binom{r-i}{g-i} \cdot 1_{\wedge^{g-1} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^g V} \\
M & \xrightarrow{(-1)^{g-i} i \cdot \varphi_{g-i,i-1}^{13} \oplus (-1)^{i(g-i-1)} (g-i) \cdot \varphi_{i,g-i-1}^{13}} & \wedge^{g-1} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}
\end{array}$$

where

$$M = \wedge^{g-i} V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^{i-1} V^\vee \otimes \wedge^g V^{\vee\vee} \oplus \wedge^i V^\vee \otimes \wedge^g V^{\vee\vee} \otimes \wedge^{g-i-1} V^\vee \otimes \wedge^g V^{\vee\vee}$$

and

$$\begin{array}{ccc}
\wedge^i V^\vee \otimes \wedge^{g-i} V^\vee \otimes V^\vee & \xrightarrow{\tau_{\wedge^i V^\vee \otimes \wedge^{g-i} V^\vee, V^\vee}} & V^\vee \otimes \wedge^i V^\vee \otimes \wedge^{g-i} V^\vee \\
\downarrow (1_{\wedge^i V^\vee} \otimes \varphi_{g-i,1}, (1_{\wedge^{g-i} V^\vee} \otimes \varphi_{i,1}) \circ (\tau_{\wedge^i V^\vee, \wedge^{g-i} V^\vee} \otimes 1_{V^\vee})) & & \downarrow D_{1,g} \otimes \varphi_{i,g-i} \\
\wedge^i V^\vee \otimes \wedge^{g-i+1} V^\vee \oplus \wedge^{g-i} V^\vee \otimes \wedge^{i+1} V^\vee & & \wedge^{g-1} V^\vee \otimes \wedge^g V^\vee \otimes \wedge^g V^\vee \\
\downarrow D_{i,g} \otimes D_{g-i+1,g} \oplus D_{g-i,g} \otimes D_{i+1,g} & & \downarrow r_{\wedge^g V} g \binom{g}{g-i}^{-1} \binom{r-i}{g-i} \cdot 1_{\wedge^{g-1} V^\vee \otimes \wedge^g V^\vee \otimes \wedge^g V^\vee} \\
\wedge^{g-i} V^\vee \otimes \wedge^g V^\vee \otimes \wedge^{i-1} V^\vee \otimes \wedge^g V^\vee \oplus \wedge^i V^\vee \otimes \wedge^g V^\vee \otimes \wedge^{g-i-1} V^\vee \otimes \wedge^g V^\vee & \xrightarrow{(-1)^{g-i} i \cdot \varphi_{g-i,i-1}^{13} \oplus (-1)^{i(g-i-1)} (g-i) \cdot \varphi_{i,g-i-1}^{13}} & \wedge^{g-1} V^\vee \otimes \wedge^g V^\vee \otimes \wedge^g V^\vee.
\end{array}$$

Proof. We first apply Corollary 3.8 with

$$g = \iota_{1,g} : V \rightarrow \text{hom}(\wedge^g V^\vee, \wedge^{g-1} V^\vee) \text{ and } (S, X, Y) = (\wedge^i V, \wedge^{g-i} V, \wedge^g V).$$

Since $r_{\wedge^g V} = r_{\wedge^g V}^{-1}$, the result is that, setting $\mu := \binom{g}{g-i}^{-1} \binom{r-i}{g-i}$,

$$\begin{aligned}
& r_{\wedge^g V} \mu \cdot (1_{\wedge^{g-1} V^\vee \otimes \wedge^g V^{\vee\vee}} \otimes i_{\wedge^g V}) \circ (D^{1,g} \otimes \varphi_{i,g-i}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \\
&= \left(\varphi_{\iota_{1,g}} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}} \right) \circ (1_V \otimes \varphi_{g-i,g}^{13}) \circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V}. \tag{34}
\end{aligned}$$

Here we recall that, by definition,

$$\varphi_{g-i,g}^{13} := (\varphi_{g-i,i} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \circ (1_{\wedge^{g-i} V^\vee} \otimes \tau_{\wedge^g V^{\vee\vee}, \wedge^i V^\vee} \otimes 1_{\wedge^g V^{\vee\vee}}).$$

Hence we find

$$\begin{aligned} g \cdot (\varphi_{\iota_{1,g}} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \circ (1_V \otimes \varphi_{g-i,g}^{13}) &= g \cdot (\varphi_{\iota_{1,g}} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \\ &\circ (1_V \otimes \varphi_{g-i,i} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \circ (1_{V \otimes \wedge^{g-i} V^\vee} \otimes \tau_{\wedge^g V^{\vee\vee}, \wedge^i V^\vee} \otimes 1_{\wedge^g V^{\vee\vee}}) \\ &= g \cdot (\varphi_{\iota_{1,g}} \circ (1_V \otimes \varphi_{g-i,i}) \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \circ (1_{V \otimes \wedge^{g-i} V^\vee} \otimes \tau_{\wedge^g V^{\vee\vee}, \wedge^i V^\vee} \otimes 1_{\wedge^g V^{\vee\vee}}) = a + b \end{aligned} \quad (35)$$

where

$$\begin{aligned} a &:= (g-i) \cdot (\varphi_{g-i-1,i} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \circ (\varphi_{\iota_{1,g-i}} \otimes \tau_{\wedge^g V^{\vee\vee}, \wedge^i V^\vee} \otimes 1_{\wedge^g V^{\vee\vee}}), \\ b &:= (-1)^{g-i} i \cdot (\varphi_{g-i,i-1} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \circ (1_{\wedge^{g-i} V^\vee} \otimes \varphi_{\iota_{1,i}} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \\ &\quad \circ (\tau_{V, \wedge^{g-i} V^\vee} \otimes \tau_{\wedge^g V^{\vee\vee}, \wedge^i V^\vee} \otimes 1_{\wedge^g V^{\vee\vee}}) \end{aligned}$$

and where we have use the equality

$$\begin{aligned} g \cdot \varphi_{\iota_{1,g}} \circ (1_V \otimes \varphi_{g-i,i}) &= (g-i) \cdot \varphi_{g-i-1,i} \circ (\varphi_{\iota_{1,g-i}} \otimes 1_{\wedge^i V^\vee}) \\ &\quad + (-1)^{g-i} i \cdot \varphi_{g-i,i-1} \circ (1_{\wedge^{g-i} V^\vee} \otimes \varphi_{\iota_{1,i}}) \circ (\tau_{V, \wedge^{g-i} V^\vee} \otimes 1_{\wedge^i V^\vee}) \end{aligned}$$

of Proposition 5.2 at the end. Inserting (35) in (34) yields

$$\begin{aligned} r_{\wedge^g V} \mu g \cdot (D^{1,g} \otimes \varphi_{i,g-i}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} &= a \circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \\ &\quad + b \circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V}. \end{aligned} \quad (36)$$

We now compute $a \circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V}$, using the following formulas:

$$\begin{aligned} D^{i,g} \otimes D^{g-i,g} &= (\varphi_{\iota_{i,g}} \otimes 1_{\wedge^g V^{\vee\vee}} \otimes \varphi_{\iota_{g-i,g}} \otimes 1_{\wedge^g V^{\vee\vee}}) \circ (1_{\wedge^i V} \otimes C_{\wedge^g V^\vee} \otimes 1_{\wedge^{g-i} V} \otimes C_{\wedge^g V^\vee}) \quad (\text{by (27)}) \end{aligned} \quad (37)$$

$$\begin{aligned} &(\varphi_{g-i-1,i} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \circ (1_{\wedge^{g-i-1} V^\vee} \otimes \tau_{\wedge^g V^{\vee\vee}, \wedge^i V^\vee} \otimes 1_{\wedge^g V^{\vee\vee}}) \\ &= \varphi_{g-i-1,i}^{13} \quad (\text{by definition}) \end{aligned} \quad (38)$$

$$\begin{aligned} \varphi_{\iota_{1,g-i}} \circ (1_V \otimes \varphi_{\iota_{i,g}}) &= \varphi_{\iota_{i+1,g}} \circ (\varphi_{i,1}^\tau \otimes 1_{\wedge^g V^\vee}) \\ &= (-1)^i \cdot \varphi_{\iota_{i+1,g}} \circ (\varphi_{1,i} \otimes 1_{\wedge^g V^\vee}) \quad (\text{by Prop. 4.1 (2)}) \end{aligned} \quad (39)$$

$$\begin{aligned} &(\varphi_{\iota_{i+1,g}} \otimes 1_{\wedge^g V^{\vee\vee}} \otimes \varphi_{\iota_{g-i,g}} \otimes 1_{\wedge^g V^{\vee\vee}}) \circ (1_{\wedge^{i+1} V} \otimes C_{\wedge^g V^\vee} \otimes 1_{\wedge^{g-i} V} \otimes C_{\wedge^g V^\vee}) \\ &= D^{i+1,g} \otimes D^{g-i,g} \quad (\text{by (27)}). \end{aligned} \quad (40)$$

We have

$$\begin{aligned}
a \circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} &= (g-i) \cdot (\varphi_{g-i-1,i} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \\
&\circ (\varphi_{\iota_{1,g-i}} \otimes \tau_{\wedge^g V^{\vee\vee}, \wedge^i V^{\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \text{ (by (37))} \\
&= (g-i) \cdot (\varphi_{g-i-1,i} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \circ (1_{\wedge^{g-i-1} V^{\vee}} \otimes \tau_{\wedge^g V^{\vee\vee}, \wedge^i V^{\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \\
&\circ (\varphi_{\iota_{1,g-i}} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^i V^{\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \circ (1_V \otimes \varphi_{\iota_{i,g}} \otimes 1_{\wedge^g V^{\vee\vee}} \otimes \varphi_{\iota_{g-i,g}} \otimes 1_{\wedge^g V^{\vee\vee}}) \\
&\circ (1_V \otimes \wedge^i V \otimes C_{\wedge^g V^{\vee}} \otimes 1_{\wedge^{g-i} V} \otimes C_{\wedge^g V^{\vee}}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \text{ (by (38))} \\
&= (g-i) \cdot \varphi_{g-i-1,i}^{13} \circ (\varphi_{\iota_{1,g-i}} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^i V^{\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \circ (1_V \otimes \varphi_{\iota_{i,g}} \otimes 1_{\wedge^g V^{\vee\vee}} \otimes \varphi_{\iota_{g-i,g}} \otimes 1_{\wedge^g V^{\vee\vee}}) \\
&\circ (1_V \otimes \wedge^i V \otimes C_{\wedge^g V^{\vee}} \otimes 1_{\wedge^{g-i} V} \otimes C_{\wedge^g V^{\vee}}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \text{ (by (39))} \\
&= (-1)^i (g-i) \cdot \varphi_{g-i-1,i}^{13} \circ (\varphi_{\iota_{i+1,g}} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^i V^{\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \\
&\circ (\varphi_{1,i} \otimes 1_{\wedge^g V^{\vee} \otimes \wedge^g V^{\vee\vee}} \otimes \varphi_{\iota_{g-i,g}} \otimes 1_{\wedge^g V^{\vee\vee}}) \\
&\circ (1_V \otimes \wedge^i V \otimes C_{\wedge^g V^{\vee}} \otimes 1_{\wedge^{g-i} V} \otimes C_{\wedge^g V^{\vee}}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \\
&= (-1)^i (g-i) \cdot \varphi_{g-i-1,i}^{13} \circ (\varphi_{\iota_{i+1,g}} \otimes 1_{\wedge^g V^{\vee\vee}} \otimes \varphi_{\iota_{g-i,g}} \otimes 1_{\wedge^g V^{\vee\vee}}) \\
&\circ (1_{\wedge^{i+1} V} \otimes C_{\wedge^g V^{\vee}} \otimes 1_{\wedge^{g-i} V} \otimes C_{\wedge^g V^{\vee}}) \\
&\circ (\varphi_{1,i} \otimes 1_{\wedge^{g-i} V}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \text{ (by (40))} \\
&= (-1)^i (g-i) \cdot \varphi_{g-i-1,i}^{13} \circ (D^{i+1,g} \otimes D^{g-i,g}) \circ (\varphi_{1,i} \otimes 1_{\wedge^{g-i} V}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \tag{41}
\end{aligned}$$

We compute $b \circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V}$, using the following formulas:

$$\begin{aligned}
&\varphi_{g-i,i-1} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}} \\
&= \varphi_{g-i,i-1}^{13} \circ (1_{\wedge^{g-i} V^{\vee}} \otimes \tau_{\wedge^{i-1} V^{\vee}, \wedge^g V^{\vee\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \text{ (by definition)} \tag{42}
\end{aligned}$$

$$\begin{aligned}
&\varphi_{\iota_{1,i}} \circ (1_V \otimes \varphi_{\iota_{g-i,g}}) = \varphi_{\iota_{g-i+1,g}} \circ (\varphi_{g-i,1}^{\tau} \otimes 1_{\wedge^g V^{\vee}}) \\
&= (-1)^{g-i} \cdot \varphi_{\iota_{g-i+1,g}} \circ (\varphi_{1,g-i} \otimes 1_{\wedge^g V^{\vee}}) \text{ (by Prop. 4.1 (2))} \tag{43}
\end{aligned}$$

$$\begin{aligned}
&(\varphi_{\iota_{i,g}} \otimes 1_{\wedge^g V^{\vee\vee}} \otimes \varphi_{\iota_{g-i+1,g}} \otimes 1_{\wedge^g V^{\vee\vee}}) \circ (1_{\wedge^i V} \otimes C_{\wedge^g V^{\vee}} \otimes 1_{\wedge^{g-i+1} V} \otimes C_{\wedge^g V^{\vee}}) \\
&= D^{i,g} \otimes D^{g-i+1,g} \text{ (by (27))}, \tag{44}
\end{aligned}$$

together with the following equality, which is the consequence of a boring computation involving the functoriality of the \otimes -operation, that of the τ -constraint and the anti-commutativity constraint in the alternating algebra:

$$\begin{aligned}
&(1_{\wedge^{g-i} V^{\vee}} \otimes \tau_{\wedge^{i-1} V^{\vee}, \wedge^g V^{\vee\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \circ (1_{\wedge^{g-i} V^{\vee}} \otimes \varphi_{\iota_{g-i+1,g}} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \\
&\circ (\varphi_{\iota_{i,g}} \otimes \varphi_{1,g-i} \otimes 1_{\wedge^g V^{\vee} \otimes \wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \circ (\tau_{V, \wedge^i V \otimes \wedge^g V^{\vee}} \otimes \tau_{\wedge^g V^{\vee\vee}, \wedge^{g-i} V \otimes \wedge^g V^{\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \\
&\circ (1_V \otimes \wedge^i V \otimes C_{\wedge^g V^{\vee}} \otimes 1_{\wedge^{g-i} V} \otimes C_{\wedge^g V^{\vee}}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \\
&= (-1)^{g-i} (\varphi_{\iota_{i,g}} \otimes 1_{\wedge^g V^{\vee\vee}} \otimes \varphi_{\iota_{g-i+1,g}} \otimes 1_{\wedge^g V^{\vee\vee}}) \circ (1_{\wedge^i V} \otimes C_{\wedge^g V^{\vee}} \otimes 1_{\wedge^{g-i+1} V} \otimes C_{\wedge^g V^{\vee}}) \\
&\circ (1_{\wedge^i V^{\vee}} \otimes \varphi_{g-i,1}). \tag{45}
\end{aligned}$$

We have

$$\begin{aligned}
& b \circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} = (-1)^{g-i} i \cdot (\varphi_{g-i, i-1} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \\
& \quad \circ (1_{\wedge^{g-i} V^{\vee}} \otimes \varphi_{\iota_{1,i}} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \circ (\tau_{V, \wedge^{g-i} V^{\vee}} \otimes \tau_{\wedge^g V^{\vee\vee}, \wedge^i V^{\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \\
& \quad \circ (1_V \otimes D^{i,g} \otimes D^{g-i,g}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \text{ (by (37))} \\
& = (-1)^{g-i} i \cdot (\varphi_{g-i, i-1} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \circ (1_{\wedge^{g-i} V^{\vee}} \otimes \varphi_{\iota_{1,i}} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \\
& \quad \circ (\tau_{V, \wedge^{g-i} V^{\vee}} \otimes \tau_{\wedge^g V^{\vee\vee}, \wedge^i V^{\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \circ (1_V \otimes \varphi_{\iota_{i,g}} \otimes 1_{\wedge^g V^{\vee\vee}} \otimes \varphi_{\iota_{g-i,g}} \otimes 1_{\wedge^g V^{\vee\vee}}) \\
& \quad \circ (1_{V \otimes \wedge^i V} \otimes C_{\wedge^g V^{\vee}} \otimes 1_{\wedge^{g-i} V} \otimes C_{\wedge^g V^{\vee}}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \text{ (by (42))} \\
& = (-1)^{g-i} i \cdot \varphi_{g-i, i-1}^{13} \circ (1_{\wedge^{g-i} V^{\vee}} \otimes \tau_{\wedge^{i-1} V^{\vee}, \wedge^g V^{\vee\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \circ (1_{\wedge^{g-i} V^{\vee}} \otimes \varphi_{\iota_{1,i}} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \\
& \quad \circ (\tau_{V, \wedge^{g-i} V^{\vee}} \otimes \tau_{\wedge^g V^{\vee\vee}, \wedge^i V^{\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \circ (1_V \otimes \varphi_{\iota_{i,g}} \otimes 1_{\wedge^g V^{\vee\vee}} \otimes \varphi_{\iota_{g-i,g}} \otimes 1_{\wedge^g V^{\vee\vee}}) \\
& \quad \circ (1_{V \otimes \wedge^i V} \otimes C_{\wedge^g V^{\vee}} \otimes 1_{\wedge^{g-i} V} \otimes C_{\wedge^g V^{\vee}}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \\
& = (-1)^{g-i} i \cdot \varphi_{g-i, i-1}^{13} \circ (1_{\wedge^{g-i} V^{\vee}} \otimes \tau_{\wedge^{i-1} V^{\vee}, \wedge^g V^{\vee\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \circ (1_{\wedge^{g-i} V^{\vee}} \otimes \varphi_{\iota_{1,i}} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \\
& \quad \circ (\varphi_{\iota_{i,g}} \otimes 1_V \otimes \varphi_{\iota_{g-i,g}} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \circ (\tau_{V, \wedge^i V \otimes \wedge^g V^{\vee}} \otimes \tau_{\wedge^g V^{\vee\vee}, \wedge^{g-i} V \otimes \wedge^g V^{\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \\
& \quad \circ (1_{V \otimes \wedge^i V} \otimes C_{\wedge^g V^{\vee}} \otimes 1_{\wedge^{g-i} V} \otimes C_{\wedge^g V^{\vee}}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \text{ (by (43))} \\
& = i \cdot \varphi_{g-i, i-1}^{13} \circ (1_{\wedge^{g-i} V^{\vee}} \otimes \tau_{\wedge^{i-1} V^{\vee}, \wedge^g V^{\vee\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \circ (1_{\wedge^{g-i} V^{\vee}} \otimes \varphi_{\iota_{g-i+1,g}} \otimes 1_{\wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \\
& \quad \circ (\varphi_{\iota_{i,g}} \otimes \varphi_{1, g-i} \otimes 1_{\wedge^g V^{\vee} \otimes \wedge^g V^{\vee\vee} \otimes \wedge^g V^{\vee\vee}}) \circ (\tau_{V, \wedge^i V \otimes \wedge^g V^{\vee}} \otimes \tau_{\wedge^g V^{\vee\vee}, \wedge^{g-i} V \otimes \wedge^g V^{\vee}} \otimes 1_{\wedge^g V^{\vee\vee}}) \\
& \quad \circ (1_{V \otimes \wedge^i V} \otimes C_{\wedge^g V^{\vee}} \otimes 1_{\wedge^{g-i} V} \otimes C_{\wedge^g V^{\vee}}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \text{ (by (45))} \\
& = (-1)^{g-i} i \cdot \varphi_{g-i, i-1}^{13} \circ (\varphi_{\iota_{i,g}} \otimes 1_{\wedge^g V^{\vee\vee}} \otimes \varphi_{\iota_{g-i+1,g}} \otimes 1_{\wedge^g V^{\vee\vee}}) \\
& \quad \circ (1_{\wedge^i V} \otimes C_{\wedge^g V^{\vee}} \otimes 1_{\wedge^{g-i+1} V} \otimes C_{\wedge^g V^{\vee}}) \circ (1_{\wedge^i V^{\vee}} \otimes \varphi_{g-i, 1}) \text{ (by (44))} \\
& = (-1)^{g-i} i \cdot \varphi_{g-i, i-1}^{13} \circ (D^{i,g} \otimes D^{g-i+1,g}) \circ (1_{\wedge^i V^{\vee}} \otimes \varphi_{g-i, 1}). \tag{46}
\end{aligned}$$

Inserting (41) and (46) in (36) gives

$$\begin{aligned}
& r_{\wedge^g V} \mu g \cdot (D^{1,g} \otimes \varphi_{i, g-i}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \\
& = (-1)^i (g-i) \cdot \varphi_{g-i-1, i}^{13} \circ (D^{i+1,g} \otimes D^{g-i,g}) \circ (\varphi_{1,i} \otimes 1_{\wedge^{g-i} V}) \circ \tau_{\wedge^i V \otimes \wedge^{g-i} V, V} \\
& \quad + (-1)^{g-i} i \cdot \varphi_{g-i, i-1}^{13} \circ (D^{i,g} \otimes D^{g-i+1,g}) \circ (1_{\wedge^i V^{\vee}} \otimes \varphi_{g-i, 1}). \tag{47}
\end{aligned}$$

Another computation involving the functoriality of the \otimes -operation, that of the τ -constraint and the anti-commutativity constraint in the alternating algebra reveals that:

$$\begin{aligned}
& (-1)^i (g-i) \cdot \varphi_{g-i-1, i}^{13} \circ (D^{i+1,g} \otimes D^{g-i,g}) \circ (\varphi_{1,i} \otimes 1_{\wedge^{g-i} V}) \circ \tau_{\wedge^i V^{\vee} \otimes \wedge^{g-i} V^{\vee}, V^{\vee}} \\
& = (-1)^{i(g-i-1)} (g-i) \cdot \varphi_{i, g-i-1}^{13} \circ (D^{g-i,g} \otimes D^{i+1,g}) \circ (1_{\wedge^{g-i} V} \otimes \varphi_{i, 1}) \circ (\tau_{\wedge^i V, \wedge^{g-i} V} \otimes 1_V). \tag{48}
\end{aligned}$$

The commutativity of the first diagram now follows from (47) and (48).

The second commutative diagram is obtained in a similar way, starting with Corollary 3.8 applied with $h = \iota_{1,g}^* : V^{\vee} \rightarrow \text{hom}(\wedge^g V, \wedge^{g-1} V)$ and $(S, X, Y) = (\wedge^i V, \wedge^{g-i} V, \wedge^g V)$ and employing the appropriate dual statements. \square

5.1. Proof of Lemma 5.4. The proof of Lemma 5.4 will be divided in several steps. We will use the shorthand $C_p := C_{\wedge^p V} : \mathbb{I} \rightarrow \wedge_p^p V$ in the sequel.

Step 1

We claim the commutativity of the following diagrams for every $m \geq 1$:

$$\begin{array}{ccc}
\Lambda_0^1 V \otimes \Lambda_1^1 V \otimes \Lambda_{m-1}^{m-1} V & \xrightarrow{\delta_{0,1}^{1,1} \otimes 1_{\Lambda_{m-1}^{m-1} V}} & \Lambda_0^1 V \otimes \Lambda_{m-1}^{m-1} V \\
\uparrow 1_V \otimes C_1 \otimes C_{m-1} & & \downarrow \varphi_{0,m-1}^{1,m-1} \\
V & \xrightarrow{1_V \otimes C_m} \Lambda_0^1 V \otimes \Lambda_m^m V & \xrightarrow{\delta_{0,m}^{1,m}} \Lambda_{m-1}^m V
\end{array}
\quad
\begin{array}{ccc}
\Lambda_1^0 V \otimes \Lambda_1^1 V \otimes \Lambda_{m-1}^{m-1} V & \xrightarrow{\delta_{1,1}^{0,1} \otimes 1_{\Lambda_{m-1}^{m-1} V}} & \Lambda_1^0 V \otimes \Lambda_{m-1}^{m-1} V \\
\uparrow 1_V \vee \otimes C_1 \otimes C_{m-1} & & \downarrow \varphi_{1,m-1}^{0,m-1} \\
V^V & \xrightarrow{1_V \vee \otimes C_m} \Lambda_1^0 V \otimes \Lambda_m^m V & \xrightarrow{\delta_{1,m}^{0,m}} \Lambda_m^{m-1} V.
\end{array}
\tag{49}$$

The proof of the commutativity of the second diagram is identical to that of the first one, so we will concentrate on the first. The case $m = 1$ is trivial and the general case is done by induction, assuming it true for m .

We will first need a simple lemma, whose proof is just an application of Lemma 2.8 (3), Lemma 2.8 (4) and (31).

Lemma 5.8. *The following diagram is commutative, for every $p \geq 0$,*

$$\begin{array}{ccc}
& \Lambda_p^p V \otimes \Lambda_1^1 V & \\
C_p \otimes C_1 \nearrow & \downarrow \varphi_{p,1}^{p,1} & \\
\mathbb{I} \xrightarrow{C_{p+1}} & \Lambda_{p+1}^{p+1} V &
\end{array}$$

We now consider the following diagram, where

$$f := m \cdot \delta_{0,m}^{1,m} \otimes 1_{\Lambda_1^1 V} \text{ and } g := (-1)^m \cdot \left(1_{\Lambda_m^m V} \otimes \delta_{0,1}^{1,1} \right) \circ \left(\tau_{\Lambda_0^1 V, \Lambda_m^m V} \otimes 1_{\Lambda_1^1 V} \right):$$

$$\begin{array}{ccc}
\Lambda_0^1 V \otimes \Lambda_m^m V \otimes \Lambda_1^1 V & \xrightarrow{(f,g)} & \Lambda_{m-1}^m V \otimes \Lambda_1^1 V \oplus \Lambda_m^m V \otimes \Lambda_0^1 V \\
\uparrow 1_V \otimes C_m \otimes C_1 & \downarrow \varphi_{m,1}^{m,1} & \downarrow \varphi_{m-1,1}^{m,1} \oplus \varphi_{m,0}^{m,1} \\
V & \xrightarrow{1_V \otimes C_{m+1}} \Lambda_0^1 V \otimes \Lambda_{m+1}^{m+1} V & \xrightarrow{(m+1) \cdot \delta_{0,m+1}^{1,m+1}} \Lambda_m^{m+1} V.
\end{array}
\tag{B}$$

The region (A) commutes by Lemma 5.8 and the region (B) is commutative thanks to the first diagram of Corollary 5.3. We deduce that we have:

$$(m+1) \cdot \delta_{0,m+1}^{1,m+1} \circ (1_V \otimes C_{m+1}) = \left(1_{\Lambda_{m+1}^{m+1} V} \oplus 1_{\Lambda_m^{m+1} V} \right) \circ (a, b) = a + b, \tag{50}$$

where

$$\begin{aligned}
a &= m \cdot \varphi_{m-1,1}^{m,1} \circ \left(\delta_{0,m}^{1,m} \otimes 1_{\Lambda_1^1 V} \right) \circ (1_V \otimes C_m \otimes C_1), \\
b &= (-1)^m \cdot \varphi_{m,0}^{m,1} \circ \left(1_{\Lambda_m^m V} \otimes \delta_{0,1}^{1,1} \right) \circ \left(\tau_{\Lambda_0^1 V, \Lambda_m^m V} \otimes 1_{\Lambda_1^1 V} \right) \circ (1_V \otimes C_m \otimes C_1).
\end{aligned}$$

We will now derive an alternative expression for a by looking at the following diagram:

$$\begin{array}{ccccc}
& \Lambda_0^1 V \otimes \Lambda_m^m V \otimes \Lambda_1^1 V & \xrightarrow{\delta_{0,m}^{1,m} \otimes 1_{\Lambda_1^1 V}} & \Lambda_{m-1}^m V \otimes \Lambda_1^1 V & \\
& \uparrow 1_V \otimes C_m \otimes C_1 & & \uparrow \varphi_{0,m-1}^{1,m-1} \otimes 1_{\Lambda_1^1 V} & \downarrow \varphi_{m-1,1}^{m,1} \\
& \Lambda_0^1 V \otimes \Lambda_1^1 V \otimes \Lambda_{m-1}^{m-1} V & \xrightarrow{\delta_{0,1}^{1,1} \otimes 1_{\Lambda_{m-1}^{m-1} V} \otimes \Lambda_1^1 V} & \Lambda_0^1 V \otimes \Lambda_{m-1}^{m-1} V \otimes \Lambda_1^1 V & \xrightarrow{\varphi_{0,m-1}^{1,m-1}} \Lambda_m^{m+1} V \\
\uparrow 1_V \otimes C_1 \otimes C_{m-1} \otimes C_1 & \downarrow 1_{\Lambda_0^1 V} \otimes \varphi_{m-1,1}^{m-1,1} & & \downarrow 1_{\Lambda_0^1 V} \otimes \varphi_{m-1,1}^{m-1,1} & \uparrow \varphi_{0,m}^{1,m} \\
V & \xrightarrow{1_V \otimes C_m} \Lambda_0^1 V \otimes \Lambda_m^m V & \xrightarrow{\delta_{0,1}^{1,1} \otimes 1_{\Lambda_m^m V}} & \Lambda_0^1 V \otimes \Lambda_m^m V & \\
& \downarrow 1_{\Lambda_0^1 V} \otimes \varphi_{m-1,1}^{m-1,1} & & \downarrow 1_{\Lambda_0^1 V} \otimes \varphi_{m-1,1}^{m-1,1} & \\
& \Lambda_0^1 V \otimes \Lambda_1^1 V \otimes \Lambda_m^m V & \xrightarrow{\delta_{0,1}^{1,1} \otimes 1_{\Lambda_m^m V}} & \Lambda_0^1 V \otimes \Lambda_m^m V &
\end{array}
\tag{A} \tag{B} \tag{C} \tag{D} \tag{E}$$

Here the region (A) is again commutative by Lemma 5.8, the region (C) is commutative by our induction assumption (49) and the functoriality of \otimes and (D) commutes by the associativity constraint. We deduce

$$a = m \cdot \varphi_{0,m}^{1,m} \circ \left(\delta_{0,1}^{1,1} \otimes 1_{\Lambda_m^m V} \right) \circ (1_V \otimes C_1 \otimes C_m). \tag{51}$$

We now compute b , noticing that $\tau_{\wedge_0^1 V, \wedge_m^m V} \otimes 1_{\wedge_1^1 V} = \tau_{\wedge_0^1 V \otimes \wedge_1^1 V, \wedge_m^m V} \circ (1_{\wedge_0^1 V} \otimes \tau_{\wedge_m^m V, \wedge_1^1 V})$ in the first of the subsequent equalities, employing the functoriality of τ in the second one and finally appealing to the commutativity constraint $\varphi_{m,0}^{m,1} \circ \tau_{\wedge_0^1 V, \wedge_m^m V} = (-1)^m \varphi_{0,m}^{1,m}$ in third equality:

$$\begin{aligned} b &= (-1)^m \cdot \varphi_{m,0}^{m,1} \circ (1_{\wedge_m^m V} \otimes \delta_{0,1}^{1,1}) \circ \tau_{\wedge_0^1 V \otimes \wedge_1^1 V, \wedge_m^m V} \circ (1_{\wedge_0^1 V} \otimes \tau_{\wedge_m^m V, \wedge_1^1 V}) \circ (1_V \otimes C_m \otimes C_1) \\ &= (-1)^m \cdot \varphi_{m,0}^{m,1} \circ \tau_{\wedge_0^1 V, \wedge_m^m V} \circ (\delta_{0,1}^{1,1} \otimes 1_{\wedge_m^m V}) \circ (1_V \otimes C_1 \otimes C_m) \\ &= \varphi_{0,m}^{1,m} \circ (\delta_{0,1}^{1,1} \otimes 1_{\wedge_m^m V}) \circ (1_V \otimes C_1 \otimes C_m). \end{aligned} \quad (52)$$

Inserting (51) and (52) in (50) we deduce

$$(m+1) \cdot \delta_{0,m+1}^{1,m+1} \circ (1_V \otimes C_{m+1}) = (m+1) \cdot \varphi_{0,m}^{1,m} \circ (\delta_{0,1}^{1,1} \otimes 1_{\wedge_m^m V}) \circ (1_V \otimes C_1 \otimes C_m),$$

from which the claim follows.

Step 2

Noticing that $\delta_{0,1}^{1,1} = (1_V \otimes \text{ev}_V^\tau) \circ (\tau_{V,V} \otimes 1_{V^\vee})$ and $\delta_{1,1}^{0,1} = \text{ev}_V \otimes 1_{V^\vee}$, we deduce from Lemma 2.8 (2) that we have

$$\begin{aligned} \delta_{0,1}^{1,1} \circ (1_V \otimes C_1) &= (1_V \otimes \text{ev}_V) \circ (1_V \otimes \tau_{V,V^\vee}) \circ (\tau_{V,V} \otimes 1_{V^\vee}) \circ (1_V \otimes C_V) \\ &= (1_V \otimes \text{ev}_V) \circ \tau_{V,V \otimes V^\vee} \circ (1_V \otimes C_V) = (1_V \otimes \text{ev}_V) \circ (C_V \otimes 1_V) = 1_V, \\ \delta_{1,1}^{0,1} \circ (1_V \otimes C_1) &= (\text{ev}_V \otimes 1_{V^\vee}) \circ (1_V \otimes C_V) = 1_{V^\vee}. \end{aligned}$$

Hence it follows from (49) that the following diagrams are commutative:

$$\begin{array}{ccc} V & \xrightarrow{1_V \otimes C_{m-1}} & \wedge_0^1 V \otimes \wedge_{m-1}^{m-1} V \\ \downarrow 1_V \otimes C_m & & \downarrow \varphi_{0,m-1}^{1,m-1} \\ \wedge_0^1 V \otimes \wedge_m^m V & \xrightarrow{\delta_{0,m}^{1,m}} & \wedge_{m-1}^m V, \end{array} \quad \begin{array}{ccc} V^\vee & \xrightarrow{1_{V^\vee} \otimes C_{m-1}} & \wedge_1^0 V \otimes \wedge_{m-1}^{m-1} V \\ \downarrow 1_{V^\vee} \otimes C_m & & \downarrow \varphi_{1,m-1}^{0,m-1} \\ \wedge_1^0 V \otimes \wedge_m^m V & \xrightarrow{\delta_{1,m}^{0,m}} & \wedge_m^{m-1} V. \end{array} \quad (53)$$

Step 3

Next we claim that the following diagram is commutative for every $m \geq 2$:

$$\begin{array}{ccc} \wedge_1^1 V & \xrightarrow{(1_{\wedge_1^1 V} \otimes C_{m-1}, (1-m) \cdot 1_{\wedge_1^1 V} \otimes C_{m-2})} & \wedge_1^1 V \otimes \wedge_{m-1}^{m-1} V \oplus \wedge_1^1 V \otimes \wedge_{m-2}^{m-2} V \\ \downarrow 1_{\wedge_1^1 V} \otimes C_m & & \downarrow \text{ev}_V^\tau \otimes 1_{\wedge_{m-1}^{m-1} V} \oplus \varphi_{1,m-2}^{1,m-2} \\ \wedge_1^1 V \otimes \wedge_m^m V & \xrightarrow{m \cdot \delta_{1,m}^{1,m}} & \wedge_{m-1}^{m-1} V. \end{array} \quad (54)$$

Consider the following diagram, where

$$\begin{array}{ccccc} \wedge_1^1 V & \xrightarrow{\tau_{V,V^\vee}} & V^\vee \otimes V & \xrightarrow{1_{V^\vee} \otimes C_{m-1}} & \wedge_1^0 V \otimes \wedge_0^1 V \otimes \wedge_{m-1}^{m-1} V \\ \downarrow 1_V \otimes C_m & & \downarrow 1_V \otimes C_m & & \downarrow 1_{V^\vee} \otimes \varphi_{0,m-1}^{1,m-1} \\ \wedge_0^1 V \otimes \wedge_1^0 V \otimes \wedge_m^m V & \xrightarrow{\tau_{V,V^\vee} \otimes 1_{\wedge_m^m V}} & \wedge_1^0 V \otimes \wedge_0^1 V \otimes \wedge_m^m V & \xrightarrow{1_{V^\vee} \otimes \delta_{0,m}^{1,m}} & \wedge_1^0 V \otimes \wedge_m^{m-1} V \\ & \searrow \varphi_{0,1}^{1,0} \otimes 1_{\wedge_m^m V} & & \swarrow m \cdot \delta_{1,m-1}^{0,m-1} & \downarrow \varphi_{0,m-1}^{0,m-1} \oplus \varphi_{0,m-2}^{1,m-2} \\ & \wedge_1^1 V \otimes \wedge_m^m V & \xrightarrow{m \cdot \delta_{1,m}^{1,m}} & \wedge_{m-1}^{m-1} V. & \end{array}$$

(A) (B) (C)

The region (A) is commutative by Corollary 4.2 (2) with $i = l = 1$, $j = k = 0$ and $m = n$, the region (B) = $1_{V^\vee} \otimes (53)$ is commutative by the commutativity of the first diagram in (53) and the functoriality of

\otimes and (C) is commutative by the second diagram of Corollary 5.3 with $i = 1, j = 0, k = l = m - 1$. Noticing that $\varphi_{0,1}^{1,0} = 1_{V \otimes V^\vee}$, $\varphi_{0,m-1}^{0,m-1} = 1_{\wedge_{m-1}^m V}$ and $\delta_{1,0}^{0,1} = \text{ev}_V$, we deduce the equality

$$m \cdot \delta_{1,m}^{1,m} \circ (1_{V \otimes V^\vee} \otimes C_m) = \left(1_{\wedge_{m-1}^m V} \oplus 1_{\wedge_{m-1}^{m-1} V} \right) \circ (a, b) = a + b, \quad (55)$$

where

$$\begin{aligned} a &= \left(\text{ev}_V \otimes 1_{\wedge_{m-1}^{m-1} V} \right) \circ (1_{V^\vee \otimes V} \otimes C_{m-1}) \circ \tau_{V, V^\vee} = \left(\text{ev}_V^\tau \otimes 1_{\wedge_{m-1}^{m-1} V} \right) \circ \left(1_{\wedge_1^1 V} \otimes C_{m-1} \right), \\ b &= (1-m) \cdot \varphi_{0,m-1}^{1,m-2} \circ \left(1_{\wedge_0^1 V} \otimes \delta_{1,m-1}^{0,m-1} \right) \circ \left(\tau_{\wedge_1^0 V, \wedge_0^1 V} \otimes 1_{\wedge_{m-1}^{m-1} V} \right) \circ (1_{V^\vee \otimes V} \otimes C_{m-1}) \circ \tau_{V, V^\vee} \\ &= (1-m) \cdot \varphi_{0,m-1}^{1,m-2} \circ \left(1_{\wedge_0^1 V} \otimes \delta_{1,m-1}^{0,m-1} \right) \circ (1_{V \otimes V^\vee} \otimes C_{m-1}). \end{aligned}$$

Next we remark that, by the commutativity of $1_V \otimes (53)$ (second diagram of (53) with m replaced by $m-1$), $\left(1_V \otimes \delta_{1,m-1}^{0,m-1} \right) \circ (1_{V \otimes V^\vee} \otimes C_{m-1}) = \left(1_V \otimes \varphi_{1,m-2}^{0,m-2} \right) \circ (1_{V \otimes V^\vee} \otimes C_{m-2})$ and that, by definition of the multiplication in the mixed algebra, $\varphi_{0,m-1}^{1,m-2} \circ \left(1_V \otimes \varphi_{1,m-2}^{0,m-2} \right) = \varphi_{1,m-2}^{1,m-2}$, so that

$$b = (1-m) \cdot \varphi_{1,m-2}^{1,m-2} \circ (1_{V \otimes V^\vee} \otimes C_{m-2}). \quad (56)$$

Inserting (56) in (55) we find the claimed commutativity.

Step 4

We now claim that

$$\begin{array}{ccc} \text{II} & \xrightarrow{(r-m+1) \cdot C_{m-1}} & \\ \downarrow C_1 \otimes C_m & & \\ \wedge_1^1 V \otimes \wedge_m^m V & \xrightarrow{m \cdot \delta_{1,m}^{1,m}} & \wedge_{m-1}^{m-1} V. \end{array} \quad (57)$$

is commutative for $m \geq 1$, where $r := \text{rank}(V)$. According to (54) we have, for $m \geq 2$,

$$\begin{aligned} m \cdot \delta_{1,m}^{1,m} \circ (C_1 \otimes C_m) &= m \cdot \delta_{1,m}^{1,m} \circ \left(1_{\wedge_1^1 V} \otimes C_m \right) \circ C_1 = \\ &= \left(1_{\wedge_{m-1}^{m-1} V} \oplus 1_{\wedge_{m-1}^{m-1} V} \right) \circ (a \circ C_1, b \circ C_1) \end{aligned}$$

where $a = \text{ev}_V^\tau \otimes C_{m-1}$ and $b = (1-m) \cdot \varphi_{1,m-2}^{1,m-2} \circ \left(1_{\wedge_1^1 V} \otimes C_{m-2} \right)$. We have

$$a \circ C_1 = (\text{ev}_V^\tau \otimes C_{m-1}) \circ C_1 = C_{m-1} \circ \text{ev}_V^\tau \circ C_1 = r \cdot C_{m-1},$$

because $r = \text{ev}_V^\tau \circ C_V$. On the other hand, by Lemma 5.8,

$$\begin{aligned} b \circ C_1 &= (1-m) \cdot \varphi_{1,m-2}^{1,m-2} \circ (C_1 \otimes C_{m-2}) = (1-m) \cdot \varphi_{1,m-2}^{1,m-2} \circ (C_{m-2} \otimes C_1) \\ &= (1-m) \cdot C_{m-1}. \end{aligned}$$

The claimed commutativity of (57) follows for $m \geq 2$. When $m = 1$ we have, by definition, $\delta_{1,1}^{1,1} = (\text{ev}_V \otimes \text{ev}_V^\tau) \circ (\tau_{V, V^\vee \otimes V} \otimes 1_{V^\vee})$, so that

$$\begin{aligned} \delta_{1,1}^{1,1} \circ (C_1 \otimes C_1) &= (\text{ev}_V \otimes \text{ev}_V^\tau) \circ (\tau_{V, V^\vee \otimes V} \otimes 1_{V^\vee}) \circ (C_V \otimes C_V) \\ &= \text{ev}_V^\tau \circ (\text{ev}_V \otimes 1_{V \otimes V^\vee}) \circ (\tau_{V, V^\vee \otimes V} \otimes 1_{V^\vee}) \circ (C_V \otimes 1_{V \otimes V^\vee}) \circ C_V \\ &= \text{ev}_V^\tau \circ ((\text{ev}_V \otimes 1_V) \circ \tau_{V, V^\vee \otimes V} \circ (C_V \otimes 1_V)) \otimes 1_{V^\vee} \circ C_V \\ &= \text{ev}_V^\tau \circ C_V = r, \end{aligned}$$

because $(\text{ev}_V \otimes 1_V) \circ \tau_{V, V^\vee \otimes V} \circ (C_V \otimes 1_V) = (1_V \otimes \text{ev}_V) \circ (C_V \otimes 1_V) = 1_V$ by Lemma 2.8 (2).

Step 5

We can now prove that, for $0 \leq k \leq m$, we have

$$\begin{array}{ccc} & \Downarrow & \xrightarrow{(r+k-m) \cdot C_{m-k}} \\ C_k \otimes C_m & \downarrow & \\ \wedge_k^k V \otimes \wedge_m^m V & \xrightarrow{\binom{m}{k} \cdot \delta_{k,m}^{k,m}} & \wedge_{m-k}^{m-k} V. \end{array} \quad (58)$$

When $k = 0$ the claim is reduced to a triviality: we have $C_k \otimes C_m = C_m$, $\binom{r+k-m}{k} C_{m-k} = C_m$ and $\binom{m}{k} \cdot \delta_{k,m}^{k,m} = 1_{\wedge_m^m V}$. In particular we may assume $1 \leq k \leq m$. For $k = 1$ this is precisely (57), so that we may assume that the commutativity is known for $1 \leq k \leq m$ and that we would like to prove it for $2 \leq k+1 \leq m$. Consider the following diagram

$$\begin{array}{ccccccc} & & \wedge_1^1 V & & & & \\ & \nearrow C_1 & \downarrow \text{(\textcircled{\scriptsize{C}})} & \searrow 1_{\wedge_1^1 V} \otimes C_k \otimes C_m & & & \\ C_k \otimes C_1 \otimes C_m & \xrightarrow{\delta_{k,m}^{k,m}} & \wedge_k^k V \otimes \wedge_1^1 V \otimes \wedge_m^m V & \xrightarrow{\tau_{\wedge_k^k V, \wedge_1^1 V} \otimes 1_{\wedge_m^m V}} & \wedge_1^1 V \otimes \wedge_k^k V \otimes \wedge_m^m V & \xrightarrow{\binom{m}{k} \cdot 1_{\wedge_1^1 V} \otimes \delta_{k,m}^{k,m}} & \wedge_1^1 V \otimes \wedge_{m-k}^{m-k} V \\ & \searrow C_{k+1} \otimes C_m & \downarrow \text{(\textcircled{\scriptsize{A}})} & \downarrow \text{(\textcircled{\scriptsize{B}})} & \searrow \delta_{1,m-k}^{1,m-k} & & \\ & & \wedge_{k+1}^{k+1} V \otimes \wedge_m^m V & \xrightarrow{\binom{m}{k} \cdot \delta_{k+1,m}^{k+1,m}} & \wedge_{m-k-1}^{m-k-1} V & & \end{array}$$

The region (A) is commutative by Lemma 5.8:

$$\left(\varphi_{k,1}^{k,1} \otimes 1_{\wedge_m^m V} \right) \circ (C_k \otimes C_1 \otimes C_m) = \left(\varphi_{k,1}^{k,1} \circ (C_k \otimes C_1) \right) \otimes C_m = C_{k+1} \otimes C_m.$$

The region (B) is commutative by Corollary 4.2 (2). Finally, the region (C) = $1_{\wedge_1^1 V} \otimes$ (58) is commutative by induction. We deduce

$$\binom{m}{k} \cdot \delta_{k+1,m}^{k+1,m} \circ (C_{k+1} \otimes C_m) = \binom{r+k-m}{k} \cdot \delta_{1,m-k}^{1,m-k} \circ (C_1 \otimes C_{m-k}). \quad (59)$$

We now note that we have $k+1 \leq m$ if and only if $m-k \geq 1$, so that (57) with m replaced by $m-k$ gives the equality

$$(m-k) \cdot \delta_{1,m-k}^{1,m-k} \circ (C_1 \otimes C_{m-k}) = (r-m+k+1) \cdot C_{m-k-1}. \quad (60)$$

Noticing that $\binom{m}{k+1} = \frac{m-k}{k+1} \binom{m}{k}$ we deduce, inserting (60) in (59), that we have

$$\binom{m}{k+1} \cdot \delta_{k+1,m}^{k+1,m} \circ (C_{k+1} \otimes C_m) = \frac{1}{k+1} \binom{r+k-m}{k} (r-m+k+1) \cdot C_{m-k-1}.$$

The claim follows because $\binom{r+k+1-m}{k+1} = \frac{1}{k+1} \binom{r+k-m}{k} (r-m+k+1)$.

6. A POINCARÉ DUALITY ISOMORPHISM FOR THE SYMMETRIC ALGEBRAS

In this section we suppose that \mathcal{C} is rigid, \mathbb{Q} -linear and pseudo-abelian. We consider an object $V \in \mathcal{C}$ and we apply the results on Δ -graded algebras with $A = (\vee V, \varphi_{i,j}^{V,s})$ and $A^\vee = (\vee V^\vee, \varphi_{i,j}^{V^\vee,s})$. We will use the shorter notation $i_V^p := i_{V,s}^p$, $p_V^p := p_{V,s}^p$, $e_V^p := e_{V,s}^p$, $\varphi_{i,j} = \varphi_{i,j}^V := \varphi_{i,j}^{V,s}$ and $\varphi_{i,j}^\vee := \varphi_{i,j}^{V^\vee,s}$. The same argument employed in the alternating case shows that the internal multiplication morphisms are given, for every $j \geq i$, by the composite

$$\varphi_{i,j} : \vee^i V \otimes \vee^j V^\vee \xrightarrow{i_V^i \otimes i_{V^\vee}^j} (\otimes^i V) \otimes (\otimes^j V) \xrightarrow{\text{ev}_V^{i,\tau} \otimes 1_{\otimes^{j-i} V^\vee}} \otimes^{j-i} V^\vee \xrightarrow{p_{V^\vee}^{j-i}} \vee^{j-i} V^\vee.$$

These morphisms can then be lifted to the tensor algebras as in Lemma 5.1, the only difference being that the character ε has to be replaced by the trivial character. The effect of this change is that the resulting normalized family $\iota_j := j \cdot \iota_{1,j}$ is now a derivation, rather than being an anti-derivation, i.e. it satisfies the symbolic theoretic formula

$$\iota_{j+l}(x)(\omega_j \vee \omega_l) = \iota_{j+l}(x)(\omega_j) \wedge \omega_l + \omega_j \wedge \iota_{j+l}(x)(\omega_l) \text{ for } x \in V, \omega_j \in \vee^j V^\vee \text{ and } \omega_l \in \vee^l V^\vee,$$

which has a formal diagram theoretic formulation analogue to Proposition 5.2. Then the analogue of Corollary 5.3, that we leave to the reader to precisely formulate, is just a formal consequence and the proof of Lemma 5.4, suitable modified employing the analogue of this corollary, lead to the following result.

Lemma 6.1. *Let $r := \text{rank}(V)$ be the rank of V , defined as the composite*

$$r : \mathbb{I} \xrightarrow{C_Y} V \otimes V^\vee \xrightarrow{\text{ev}_Y^\tau} \mathbb{I}.$$

For every $g \geq i$ we have the equality

$$\binom{g}{i}^{-1} \binom{r+g-1}{i} \cdot C_{\vee^{g-i}V} = \delta_{i,g}^{i,g} \circ (C_{\vee^i V} \otimes C_{\vee^g V}),$$

where, for every $k \in \mathbb{N}_{\geq 1}$,

$$\binom{T}{k} := \frac{1}{k!} T(T-1) \dots (T-k+1) \in \mathbb{Q}[T] \text{ and } \binom{T}{0} = 1.$$

As in the alternating case we may define, for every $g \geq i$, the Poincaré morphisms

$$D^{i,g} := D_{\iota_{i,g}} : \vee^i V \rightarrow \vee^{g-i} V^\vee \otimes \vee^g V^{\vee\vee} \text{ and } D_{\iota_{i,g}}^* : \vee^i V^\vee \rightarrow \vee^{g-i} V \otimes \vee^g V^\vee.$$

The following result is obtained from Lemma 6.1 in the same way as Theorem 5.5 has been obtained from Lemma 5.4 with .

Theorem 6.2. *The following diagrams are commutative, for every $g \geq i \geq 0$.*

(1)

$$\begin{array}{ccc} \vee^i V \otimes \vee^i V^\vee & \xrightarrow{\binom{g}{g-i}^{-1} \binom{r+g-1}{g-i} \text{ev}_{V,s}^{i,\tau}} & \mathbb{I} \\ \downarrow D^{i,g} \otimes D_{i,g} & \searrow & \uparrow \text{ev}_{13,24}^{\phi,\phi} \\ \vee^{g-i} V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^{g-i} V \otimes \vee^g V^\vee & \xrightarrow{\text{ev}_{13,24}^{\phi,\phi}} & \mathbb{I} \end{array}$$

(2)

$$\begin{array}{ccc} \vee^i V^\vee \otimes \vee^{g-i} V^\vee \xrightarrow{1_{\vee^i V^\vee} \otimes D_{g-i,g}} \vee^i V^\vee \otimes \vee^i V^\vee \otimes \vee^g V^\vee & & \vee^i V \otimes \vee^{g-i} V \xrightarrow{1_{\vee^i V} \otimes D_{g-i,g}} \vee^i V \otimes \vee^i V^\vee \otimes \vee^g V^{\vee\vee} \\ \downarrow D_{i,g} \otimes 1_{\vee^{g-i} V^\vee} & & \downarrow D^{i,g} \otimes 1_{\vee^{g-i} V} \\ \vee^{g-i} V \otimes \vee^g V^\vee \otimes \vee^{g-i} V^\vee \xrightarrow{\text{ev}_{13,\vee^g V^\vee}^\tau} \vee^g V^\vee & & \vee^{g-i} V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^{g-i} V \xrightarrow{\text{ev}_{13,\vee^g V^\vee}^\phi} \vee^g V^{\vee\vee} \\ \downarrow \text{ev}_{V,s}^i \otimes 1_{\vee^g V^\vee} & & \downarrow \text{ev}_{V,s}^{i,\tau} \otimes 1_{\vee^g V^{\vee\vee}} \end{array}$$

(3)

$$\begin{array}{c} \xrightarrow{\binom{g}{g-i}^{-1} \binom{r+g-1}{g-i}} \\ \vee^i V \xrightarrow{D^{i,g}} \vee^{g-i} V^\vee \otimes \vee^g V^{\vee\vee} \xrightarrow{D_{g-i,g} \otimes 1_{\vee^g V^{\vee\vee}}} \vee^i V \otimes \vee^g V^\vee \otimes \vee^g V^{\vee\vee} \xrightarrow{1_{\vee^i V} \otimes \text{ev}_{V^\vee,s}^{g,\tau}} \vee^i V \\ \text{and} \\ \vee^{g-i} V^\vee \xrightarrow{D_{g-i,g}} \vee^i V \otimes \vee^g V^\vee \xrightarrow{D^{i,g} \otimes 1_{\vee^g V^\vee}} \vee^{g-i} V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^g V^\vee \xrightarrow{1_{\vee^{g-i} V^\vee} \otimes \text{ev}_{V^\vee,s}^{g,\tau}} \vee^{g-i} V^\vee. \end{array}$$

(4)

$$\begin{array}{ccc} \vee^i V \otimes \vee^{g-i} V \xrightarrow{\varphi_{i,g-i}} \vee^g V & & \vee^i V^\vee \otimes \vee^{g-i} V^\vee \xrightarrow{\varphi_{i,g-i}} \vee^g V^\vee \\ \downarrow D^{i,g} \otimes D_{g-i,g} & & \downarrow D_{i,g} \otimes D_{g-i,g} \\ \vee^{g-i} V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^i V^\vee \otimes \vee^g V^{\vee\vee} \xrightarrow{\varphi_{g-i,i}^{13 \rightarrow \vee^g V^{\vee\vee}}} \vee^g V^{\vee\vee} & & \vee^{g-i} V \otimes \vee^g V^\vee \otimes \vee^i V \otimes \vee^g V^\vee \xrightarrow{\varphi_{g-i,i}^{13 \rightarrow \vee^g V^\vee}} \vee^g V^\vee \\ \downarrow \binom{g}{g-i}^{-1} \binom{r+g-1}{g-i} \cdot i_{\vee^g V} & & \downarrow \binom{g}{g-i}^{-1} \binom{r+g-1}{g-i} \end{array}$$

We say that V has *symmetric rank* $g \in \mathbb{N}_{\geq 1}$ if $\vee^g V$ is an invertible object and $\binom{r+g-1}{g-i}$ and $\binom{r+g-1}{i}$ are invertible for every $0 \leq i \leq g$. For example, when $\text{End}(\mathbb{I})$ is a field or $r \in \mathbb{Q}$, the second condition means that r is not a root of the polynomials $\binom{T+g-1}{g-i} \in \mathbb{Q}[T]$ and $\binom{T+g-1}{i} \in \mathbb{Q}[T]$ for every $0 \leq i \leq g$, i.e. that $r \neq 1-g, 2-g, \dots, -i$ and $r \neq 1-g, 2-g, \dots, i-g$ for every $1 \leq i \leq g$.

We say that V has *strong symmetric rank* $g \in \mathbb{N}_{\geq 1}$ if $\vee^g V$ is an invertible object and $r = -g$ (hence V has symmetric rank g). With these notations Corollary 3.6 specializes to the following result.

Corollary 6.3. *If V has symmetric rank $g \in \mathbb{N}$ then, for every $0 \leq i \leq g$, the morphisms $D^{i,g}$, $D_{g-i,g}$, $D^{g-i,g}$ and $D_{i,g}$ are isomorphisms and the multiplication maps $\varphi_{i,g-i}^V$, $\varphi_{g-i,i}^V$, $\varphi_{i,g-i}^{V^\vee}$ and $\varphi_{g-i,i}^{V^\vee}$ are perfect pairings (meaning that the associate hom valued morphisms are isomorphisms). Furthermore, when V has strong symmetric rank g , we have $\binom{r+g-1}{g-i} = (-1)^{g-i}$ and $\binom{r+g-1}{i} = (-1)^i$ in the commutative diagrams of Theorem 6.2.*

We end this section with the analogue of Proposition 5.7 in this setting. This a technical result that will be crucial for the computation of [MS]. The proof is just a copy of that of Proposition 5.7.

Proposition 6.4. *The following diagrams are commutative when $\vee^g V$ is invertible of rank $r_{\vee^g V}$ (hence $r_{\vee^g V} \in \{\pm 1\}$):*

$$\begin{array}{ccc}
\vee^i V \otimes \vee^{g-i} V \otimes V & \xrightarrow{\tau_{\vee^i V \otimes \vee^{g-i} V, V}} & V \otimes \vee^i V \otimes \vee^{g-i} V \\
\downarrow (1_{\vee^i V} \otimes \varphi_{g-i,1}, (1_{\vee^{g-i} V} \otimes \varphi_{i,1}) \circ (\tau_{\vee^i V, \vee^{g-i} V} \otimes 1_V)) & & \downarrow D^{1,g} \otimes \varphi_{i,g-i} \\
\vee^i V \otimes \vee^{g-i+1} V \oplus \vee^{g-i} V \otimes \vee^{i+1} V & & \vee^{g-1} V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^g V \\
\downarrow D^{i,g} \otimes D^{g-i+1,g} \oplus D^{g-i,g} \otimes D^{i+1,g} & & \downarrow r_{\vee^g V} g \binom{g}{g-i}^{-1} \binom{r+g-1}{g-i} \cdot 1_{\vee^{g-1} V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^g V} \\
M & \xrightarrow{i \cdot \varphi_{g-i,i-1}^{13} \oplus (g-i) \cdot \varphi_{i,g-i-1}^{13}} & \vee^{g-1} V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^g V^{\vee\vee}
\end{array}$$

where

$$M = \vee^{g-i} V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^{i-1} V^\vee \otimes \vee^g V^{\vee\vee} \oplus \vee^i V^\vee \otimes \vee^g V^{\vee\vee} \otimes \vee^{g-i-1} V^\vee \otimes \vee^g V^{\vee\vee}$$

and

$$\begin{array}{ccc}
\vee^i V^\vee \otimes \vee^{g-i} V^\vee \otimes V^\vee & \xrightarrow{\tau_{\vee^i V^\vee \otimes \vee^{g-i} V^\vee, V^\vee}} & V^\vee \otimes \vee^i V^\vee \otimes \vee^{g-i} V^\vee \\
\downarrow (1_{\vee^i V^\vee} \otimes \varphi_{g-i,1}, (1_{\vee^{g-i} V^\vee} \otimes \varphi_{i,1}) \circ (\tau_{\vee^i V^\vee, \vee^{g-i} V^\vee} \otimes 1_{V^\vee})) & & \downarrow D_{1,g} \otimes \varphi_{i,g-i} \\
\vee^i V^\vee \otimes \vee^{g-i+1} V^\vee \oplus \vee^{g-i} V^\vee \otimes \vee^{i+1} V^\vee & & \vee^{g-1} V^\vee \otimes \vee^g V^\vee \otimes \vee^g V^\vee \\
\downarrow D_{i,g} \otimes D_{g-i+1,g} \oplus D_{g-i,g} \otimes D_{i+1,g} & & \downarrow r_{\vee^g V} g \binom{g}{g-i}^{-1} \binom{r+g-1}{g-i} \cdot 1_{\vee^{g-1} V^\vee \otimes \vee^g V^\vee \otimes \vee^g V^\vee} \\
\vee^{g-i} V^\vee \otimes \vee^g V^\vee \otimes \vee^{i-1} V^\vee \otimes \vee^g V^\vee \oplus \vee^i V^\vee \otimes \vee^g V^\vee \otimes \vee^{g-i-1} V^\vee \otimes \vee^g V^\vee & \xrightarrow{i \cdot \varphi_{g-i,i-1}^{13} \oplus (g-i) \cdot \varphi_{i,g-i-1}^{13}} & \vee^{g-1} V^\vee \otimes \vee^g V^\vee \otimes \vee^g V^\vee.
\end{array}$$

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